Spatial Moran Models with Local Interactions

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Abstract

We begin by considering a genetic stepping stone model with sites on the one-dimensional lattice $n^{-1}\mathbb{Z}$. Within each site, particles are subject to Moran model interactions with selection. Between interactions, they mutate and migrate independently. If the population density is held constant while the lattice density n increases, then—under suitable parameter scalings—the migration random walks converge to Brownian motions and the limiting interactions are determined by Poisson counting processes driven by clocks proportional to the local times at zero of the distances between pairs. The result is a finite-density collection of Brownian motions with local-time Moran interactions.

We study the limiting behavior of these models as the population density increases to infinity by ordering the particles with randomly assigned "levels" in the non-negative reals $\mathbb{R}^+$. If neutral interactions are restricted to occur in only one direction, so that the higher-level particle changes its type to that of the lower-level particle, the result is an ordered model that can be extended to infinite densities.

Restricted to a given maximum level, these infinite-density ordered models have the same empirical location/type distributions as the original, symmetric Moran models. In the stepping-stone case, we establish this by means of a generator argument. In the Brownian case, we establish it through a more direct coupling.

Under appropriate initial conditions, these ordered models have a simple Poisson structure. In the Brownian case, for each t, there exists a measure-valued diffusion ν_t such that the point process consisting of the location, type, and level of each particle is conditionally Poisson with mean measure $\nu_t \times \ell_{\mathbb{R}^+}$.

We study this diffusion process for the Brownian case with selection, showing that it almost surely has continuous paths and giving a martingale characterization.

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Finally, I wish to thank my wife, Tracey, whose love and support mean more to me than even *she* suspects.

List of Notation

Page numbers in the right-hand column reference most appearances of the symbol. An italicized page number designates a definitive reference. (For standard notation, no page references are given.)

$\langle Z \rangle_t$	angle-brackets, conditional quadratic variation	
A^c	complement of A	
$X := \dots$	definition of X as ...	
$X \sim F$	distribution of X is F	
$=^{a.s.}$	equal almost surely	
$=^d$	equal in distribution	
$[Z]_t$	quadratic variation	
$\|f\|_\infty$	sup-norm $\sup_x \|f(x)\|$	
$\mathbf{x}$	vector $(x_1, x_2, \dots)$	
$\xrightarrow{a.s.}$	convergence almost surely	
$\xrightarrow{L^p}$	convergence in L_p	
$\xrightarrow{p}$	convergence in probability	
$\Rightarrow$	weak convergence	

A_n	stepping stone generator (ordered)	*23*, 28, 29
$A_{n,\lambda}$	stepping stone generator (ordered, neutral)	*29*
α	bound in location space, bound on support of location space test function $\text{supp}(f) \subseteq [-\alpha, \alpha]$	34–37, 41, *62*, *65*, *67*, 68, *75*, 77, 81, 88, 91, 92, 94, 95
$B(E)$	measurable, bounded functionals $E \to \mathbb{R}$	
B^{μ}	mutation process generator	*4*, 5, 11, 23, 26, 27, 29, 32, 71–74, 76–83, 85, 89, 90
$B^{\theta}_{n,j}$	random walk generator	*4*, 23, 26, 27, 29
$\mathfrak{B}(E)$	Borel sets of E	
c_1	mean particles hitting $[-\alpha, \alpha]$ in time T	*67*, 68, 75, 83
c_2	mean square particles hitting $[-\alpha, \alpha]$ in time T	*67*, 68, 88–90, 94
$C^2(E)$	twice continuously differentiable functions $E \to \mathbb{R}$	
$C^2_c(E)$	functions in $C^2(E)$ with compact support	
$D_E[0, \infty)$	space of càdlàg paths $[0, \infty) \to E$	
$\mathfrak{D}(A)$	domain of generator A	

E	type space, or arbitrary complete, separate metric space	
η	component substitution on vectors $\eta_j(\mathbf{x}, x_0) = (x_1, x_2, \dots, x_{j-1}, x_0, x_j, \dots)$	*4*, 23, 26, 27, 29
η_{jk}	uniform $[0, 1]$ to pick “other” type for potential selective event	*42*, 43, 45, 60, 64, 71
$\mathfrak{F}$	σ-field of all events on Ω	
$\mathfrak{F}_t^X$	filtration generated by process X up to time t	
$\mathfrak{F}_t^K$	filtration of locations/levels/types above level K	*62*, 63–66, 69, 70, 72, 74, 76
$\mathfrak{F}_t^\infty$	filtration of locations/types only	*62*, 63, 65, 66, 69, 70, 73–76
$\tilde{\Gamma}$	empirical location/type process (Moran)	*24*, 28
Γ	empirical location/type process (ordered)	*24*, 28
$\tilde{\gamma}_{jk}$	index of “other” particle when $\tilde{V}_j$ jumps	*6*, 7, 8, 18
γ_{jk}	index of “other” particle when V_j jumps	*22*, *33*, 43
$\hat{\gamma}_{jk}$	index of “other” particle when $\hat{V}_j$ jumps	*60*
h^σ	random location/type function representing selective events	*73*, 74, 76, 78, 85
$\aleph_1$	cardinality of errors in present paper	
$\tilde{\iota}_{jk}$	indicator that $\tilde{\tau}_{jk}$ was a neutral event	*6*, 7, 8, 18
ι_{jk}	indicator that τ_{jk} was a neutral event	*22*, *33*, 43, 71
$\hat{\iota}_{jk}$	indicator that $\hat{\tau}_{jk}$ was a neutral event	*60*

K	particle density (per unit space)	2, *9*, 10, *11*, 12, 19–21, 23, 24, 27, 30–37, 41, 42, 46–50, 52–54, 59, 60, 62–79, 81–86, 88–94
$k_n(\cdot)$	a C^2 approximation to the absolute value $\lvert\cdot\rvert$	*87*, 88–92
$L_t^a(X)$	local time at a of X	*11*, 32, 35, 37, 42, 47, 57, 60
L_{ij}	local time $L_t^0(X_i - X_j)$	*16*, 17, 18, 32–34, 42, 47, 55, 57, 58, 60, 82, 83, 85–88
$L_{\{i,j\}}$	local time $L_t^0(\tilde{X}_i - \tilde{X}_j)$	*47*
L_{ij}^n	an approximation to L_{ij}	*87*, 88
ℓ_A	Lebesgue measure *on* a set A	
$\ell(A)$	Lebesgue measure *of* a set A	
l_t	local time L_t^0 of a standard Brownian motion	*35*, 36, *37*, 38, 41, 82–84
$\mathfrak{L}(X)$	probability law of X	

λ	neutral event rate per unordered pair	*4*, 5, 9, 11, 17, 22, 23, 26, 27, 29, 31, 33, 42, 47, 55, 56, 58, 60, 74, 76, 82, 83, 85, 86
M	characteristic martingale for $\langle u_t, g\rangle$	*74*, 75, 76, 84
M^K	characteristic martingale for $\left\langle u_t^K, g\right\rangle$	*74*, 75, 76, 84, 85
$M_{jk}^{g,y}$	mutation process martingale for $g(Y_{jk}(y,\cdot))$	*71*, 72
M_j^g	mutation process martingale for $g(Z_j)$	*72*, 78–81, 84, 85, 89, 91
N	standard, rate 1 Poisson counting process (or Poisson random variable)	
N_{ij}^{λ}	neutral event Poisson counting process	*5*, 9, 17, 18, 21, 22, 28, 32, 33, 42, 46, 47, 52, 55, 56, 58, 60, 64
N_{ij}^{σ}	selective event Poisson counting process	8, *9*, 18, 22
$\bar{N}_{ij}^{\sigma}$	potential selective event Poisson counting process	*5*, 6, 9, 17, 18, 21, 22, 32–34, 42, 46, 47, 52, 58, 60, 61, 64

$N^K_{s,x}$	number of particles in $[x, x + \frac{1}{n}) \times [0, K]$ at time s	*93*
n_t	number n_t of particles involved in event at time $t \in \tau$	*48*, 49, 50, 52
$\mathbb{N}$	natural numbers $1, 2, 3, \ldots$	
$\mathbb{N}'$	isomorphic copy of $\mathbb{N}$ used to index particles above level K in hybrid model	*60*, 61, 64
ν_t	location/type measure-valued process	2, 32, *66*, 69, 70
ν_0	initial location/type measure on $\mathbb{R} \times E$	*11*, 12, 19, 32, 34, 42, 46, 59–62, 66
$\nu_0^{(n)}$	initial location/type measure on $n^{-1}\mathbb{Z} \times E$	*9*, 10, 12, 23, 28, 30
$\hat{\nu}_t(x, dz)$	type distribution "at" a location in real space	*70*, 74, 76, 86, 92–94
$(\Omega, \mathfrak{F}, P)$	standard probability space	
ω	sample point $\omega \in \Omega$	
P	standard probability measure on Ω	
$P(E)$	probability measures on E	
$\mathfrak{P}_1(\mathbb{N})$	finite, nonempty subsets of $\mathbb{N}$	*48*, 49, 52
$\Phi_j(t)$	permutation process for particle indices	*25*, 26, 28, 47–53, 56–58, 61, 73
$\boldsymbol{\Phi}$	vector $(\Phi_1, \Phi_2, \ldots)$	7, *25*, 26–28, 56
$\Phi(A)$	probability $P(Z \in A)$ for standard normal Z	
$\Phi(x)$	probability $P(Z < x)$ for standard normal Z	

$\phi(z_1, z_2, i, \nu)$	chooses result of reproduction event	*6*, 8, 43, 44
$\boldsymbol{\phi}_{\tilde{i}\tilde{j}}$	permutation $\boldsymbol{\phi}$ with the values i and j swapped	*26*
$\pi_{\{i,j\},k}$	random permutation $\{i, j\} \leftrightarrow \{i, j\}$	*46*, 47–52
$Poisson(\mu)$	Poisson point process with mean μ	
$\psi(\xi, \nu)$	chooses particle type for selective event	*43*, 44, 45, 62–64, 71–74, 78–81, 84, 85, 90
ψ_{jk}	"other" type picked for selective event at τ_{jk}	*43*
$\hat{\psi}_{jk}$	"other" type picked for selective event at $\hat{\tau}_{jk}$	*60*
Q	mutation transition kernel	*5*
$\mathbb{Q}$	rational numbers	
$\mathbb{R}$	real numbers	
$\mathbb{R}^+$	non-negative real numbers	
R^K	per-particle potential selective event time change	*34*, 36, 41, 42
R^K_α	per-particle potential selective event time change from particles within radius α	*34*, 35–37, 41
ρ	random ranks of levels $\tilde{U}$ at an interaction	*49*, 50–52
sgn	left-continuous sign function	*11*, 14, 15, 35, 57, 87–89, 91
$\Sigma^T_j(t)$	influence set of particle j's type at time T	*48*, 49, 50, 52–55
$\Sigma_\star$	influence set of $\sigma^\star$	*49*, 50–52

$\sigma(z_1, z_2)$	selective event rate of type z_1 over z_2	*4*, 6–9, 11, 17, 22, 23, 26–31, 33, 42–44, 56, 58, 60, 72, 73, 78–81, 84, 85, 90
$\bar{\sigma}$	selective event rate upper bound $0 \leq \sigma \leq \bar{\sigma}$	*4*, 6–8, 17, 22, 33–37, 41, 42, 44, 45, 47, 56, 58, 60, 61, 71, 72, 78–85, 90
σ_t	particles involved in event at time $t \in \tau$	*48*, 49–52
σ'	finite subset of particles	*49*, 52
$\sigma_\star$	particles σ' plus particles involved in interactions at all times τ'	*49*
$\sigma\{\cdot\}$	σ-field generated by $\{\cdot\}$	
$\sigma^2_{s,x}$	centering term for $V^K_{s,x}$	*92*, 93, 94
$\text{supp}(f)$	support of a function f	
$T^\lambda_{\{i,j\},k}$	jump times of $V^\lambda_{\{i,j\}}$	*47*, 48, 52, 55, 64
$T^\lambda_{j,k}$	ordering of set $\{T^\lambda_{\{i,j\},k} : \forall i \neq j\}$	*47*
$\tilde{\tau}_{jk}$	jump times of $\tilde{V}_j$	*6*, 7, 8, 18
τ_{jk}	jump times of V_j	*22*, *33*, 43, 45, 64, 71, 72
$\hat{\tau}_{jk}$	jump times of $\hat{V}_j$	*60*
τ	countable set of all event times	*48*, 49, 52–55
τ_j	set of event times involving particle j	*48*
τ'	finite subset of τ	*49*, 52

$\tau_\star$	times of events considered by $\Sigma_\star$	*49*, 50, 52
τ_j^α	hitting time of $[-\alpha, \alpha]$ by X_j	*67*, 68, 75, 77–79, 81–83, 88–90, 94
θ	migration rate	*4*, 5, 9, 10, 12, 14–18, 32–35, 37, 41, 42, 47, 53–56, 58, 60, 67, 68, 72–74, 76, 78, 79, 82–87, 92
$\mathfrak{T}$	define type process in terms of neutral and potential selective events	*7*, 8, 18, 22, 28, 33, 44, 56, 58
$\mathfrak{T}'$	define type process in terms of neutral and potential selective events	*44*, 60
U_j	particle level for ordered model	*21*, 22, 29, 30, 33, 42, 43, 46–52, 56, 58, 60–70, 72–75, 77–86, 88–94
$\mathbf{U}$	vector $(U_1, U_2, \dots)$	*22*, 23, 24, 26–30, 32–34, 42, 44, 45, 58–60, 62, 67, 68

$\tilde{U}_j$	particle level process for intermediate model	*47*, *48*, 49–53, 55, 61, 64, 65, 73
$\tilde{\mathbf{U}}$	vector $(\tilde{U}_1, \tilde{U}_2, \dots)$	7, *55*, 56, 59
$\hat{U}_j$	uniformly distributed random variables	34, 35, 37, *63*, 65, 69
u_t^K	location-type measure for particles below K	*62*, 71
$\langle u_t^K, h\rangle$	integration of u_t^K against h	*62*, 65, 70, 72, 74–78, 82, 84, 85
$\langle u_t, h\rangle$	infinite-density $\lim_{K\to\infty} \frac{1}{K} \langle u_t^K, h\rangle$	*65*
$\mathbf{u}_{ij}$	vector $\mathbf{u}$ with components u_i and u_j swapped	*27*
$\tilde{\Upsilon}_n$	stepping stone model (Moran)	*9*, 10, 12, 19–21, 23, 24, 30, 31
$\tilde{\Upsilon}_0$	Brownian model (Moran)	*18*, 19, 30–32, 46, 55, 56, 59
Υ_n	stepping stone model (ordered)	21, *23*, 24, 28–30
$\Upsilon_{n,\lambda}$	stepping stone model (ordered, neutral)	*29*, 30
Υ_0	Brownian model (ordered)	31, 32, *33*, 34, 45, 46, 55, 58, 59
Υ_∞	Brownian model (ordered, infinite-density)	31, 32, *45*, 46, 59, 61, 62, 67
$\tilde{V}_{ij}^{(n),\lambda}$	neutral events (Moran, stepping stone)	*5*, 6, 7, 9–11, 13, 17

$\tilde{V}^{\lambda}_{ij}$	neutral events (Moran, Brownian)	*18*, 19, 55, 61, 73
$V^{(n),\lambda}_{ij}$	neutral events (ordered, stepping stone)	*22*, 26, 29
V^{λ}_{ij}	neutral events (ordered, Brownian)	*33*, 42, 43, *56*, 58, 60, 61, 72, 73, 78, 80–82, 84, 85, 90
$\hat{V}^{(n)}_{ij}$	half-rate filtering of $V^{(n),\lambda}_{ij}$	25, *26*
$V^{\lambda}_{\{i,j\}}$	bidirectional neutral events (per-pair, intermediate model)	*47*, 48, 55, 56, 58
$\hat{V}^{\lambda}_{ij}$	neutral events (hybrid, Brownian)	60, *61*
$\tilde{V}^{(n),\sigma}_{ij}$	selective events (Moran, stepping stone)	*8*, 9, 11
$\tilde{V}^{\sigma}_{ij}$	selective events (Moran, Brownian)	*18*, 19
$V^{(n),\sigma}_{ij}$	selective events (ordered, stepping stone)	*22*
$\bar{V}^{(n),\sigma}_{ij}$	potential selective events (per-pair, stepping stone)	*6*
$\bar{V}^{\sigma}_{ij}$	potential selective events (per-pair, Brownian)	*18*, 33, 34
$\bar{V}^{\sigma}_{j}$	potential selective events (per-particle, Brownian)	*34*, *42*, 43, 61
$\bar{V}^{\sigma,\Phi}_{ij}$	potential selective events (per-pair, Brownian, reordered for intermediate model)	*56*, 58
V_j	all events affecting j (ordered, Brownian)	*22*, 29, *33*, 43, 72
$\tilde{V}_j$	all events affecting j (Moran, Brownian)	*6*, 7, 18
$\hat{V}_j$	all events affecting j (hybrid, Brownian)	*60*
$V^{K}_{s,x}$	a second-moment expression for particles in $[x, x + \frac{1}{n}) \times [0, K]$	*92*, 93, 94
$\mathfrak{V}^{\sigma}_{j}$	random measure representation of selective events	*71*, 72, 80, 81, 90
$\hat{\mathfrak{V}}^{\sigma}_{j}$	centered version of $\mathfrak{V}^{\sigma}_{j}$	*71*, 72, 78, 79, 84, 85

$W, \hat{W}$	standard Brownian motion	*12*, 17, *32*, 34, *35*, 36, 37, *42*, *46*, 47, 52, 53, 56, *60*, 64, 67, 72, 78, 79, 84, 85
$W^{\mathbb{Z}}$	symmetric, rate 1 random walk on $\mathbb{Z}$	*5*, 9, 10, 17, 21, 28
$\mathfrak{W}$	the probability law of W	*53*, 67, 68
$\mathfrak{W}^{\mathbb{Z}}$	the probability law of $W^{\mathbb{Z}}$	*10*
$X_j^{(n)}$	particle location process (stepping stone)	*5*, 6, 8–10, 12, 13, 16, 17, 22, 24, 26, 28
$\mathbf{X}^{(n)}$	vector $(X_1^{(n)}, X_2^{(n)}, \dots)$	*5*, 7, 9–12, 21–24, 26–29
X_j	particle location process (Brownian)	10, 12, 16, 17, 19, 30, *32*, 33, 34, 42, 43, 45, 47, 56, 60, 62–67, 69, 70, 72, 74, 77–81, 83–93

$\mathbf{X}$	vector $(X_1, X_2, \dots)$	9, 10, 12, 17, 18, 23, 24, 26, 29, 30, *32*, 33, 34, 42, 44–46, 52, 56, 58, 59, 62, 67, 68
$\tilde{X}_j$	particle location for symmetric model derived from intermediate model	*47*, 53–57, 60, 61, 64, 65, 73
$\hat{X}_j$	particle locations for hybrid model	*60*
$X_{ij}^{(n)}$	difference $X_i^{(n)} - X_j^{(n)}$	*12*, 13, 17
X_{ij}	difference $X_i - X_j$	*12*, 13, 16, 57, 87–92
$\tilde{X}_{ij}$	difference $\tilde{X}_i - \tilde{X}_j$ (between symmetric particle pairs in intermediate model)	*57*
ξ	location/level/type process (Brownian, infinite-density, ordered)	*43*, 44, 59, 61, 62, 66, 69, 72–74, 78–81, 84, 85, 90
$\tilde{\xi}^K$	location/level process (Brownian, K-density, Moran)	*19*, 46, 59, 66
ξ^K	location/level process (Brownian, K-density, ordered)	*33*, 46, 59, 66
$\hat{\xi}$	location/level/type process (Brownian, infinite-density, hybrid)	*60*
Y, Y_{jk}	particle mutation processes	*5*, 6, 7, 9, 21, 22, 27, 28, 32, 42, 43, 46, 52, 58, 60, 64, 71

$Y_k^{(n)}$	number of particles at site k/n (stepping stone)	*9*, 10, 12
$\mathbf{Y}^{(n)}$	vector $(\dots, Y_{-1}^{(n)}, Y_0^{(n)}, Y_1^{(n)}, \dots)$	*9*, 10
$\tilde{Z}_j^{(n)}$	particle type process (stepping stone, Moran)	*6*, 7–9, 24, 28
$\tilde{\mathbf{Z}}^{(n)}$	vector $(\tilde{Z}_1^{(n)}, \tilde{Z}_2^{(n)}, \dots)$	5, *6*, 7–13, 18, 22, 24, 27, 28
$\tilde{Z}_j$	particle type process (Brownian, Moran)	*18*, 19, 58, 61, 64, 65, 73
$\tilde{\mathbf{Z}}$	vector $(\tilde{Z}_1, \tilde{Z}_2, \dots)$	*18*, 56, 59
$Z_j^{(n)}$	particle type process (stepping stone, ordered)	*22*, 24, 28, 29
$\mathbf{Z}^{(n)}$	vector $(Z_1^{(n)}, Z_2^{(n)}, \dots)$	21, *22*, 23, 24, 26–29
Z_j	particle type process (Brownian, ordered)	30, *33*, 43, 58, 61–66, 69, 70, 72–74, 77–81, 84, 85, 89–93
$\mathbf{Z}$	vector $(Z_1, Z_2, \dots)$	22, 30, 32, *33*, 34, 42–45, 58, 59, 61, 62
$\hat{Z}_j$	particle type process (Brownian, hybrid)	*60*, 61
$\hat{\mathbf{Z}}$	collection $(\hat{Z}_j : j \in \mathbb{N} \uplus \mathbb{N}')$	*60*
$\mathbb{Z}$	integers $\dots, -1, 0, 1, \dots$	
$\mathbb{Z}^+$	non-negative integers $0, 1, 2, \dots$	
ζ_{jk}	uniform on $[0, 1]$ to determine when jump in $\bar{V}_j^\sigma$ becomes actual event	*5*, 6–9, 18, 46, 52, 60

Chapter 1

Spatial Moran Models

"Interestingly, according to modern astronomers, space is finite. This is a very comforting thought—particularly for people who can never remember where they have left things."

— *Woody Allen (b. 1935)*

1.1 Introduction

We wish to model the spread of genetic innovation through spatially structured populations. In this paper, we present a number of monoecious, haploid particle models that incorporate explicit one-dimensional spatial structure and local reproductive interactions between particles with selection.

In this chapter, we consider Moran models with spatial structure and selection. We begin, in the next section, by briefly considering the standard Moran model, a finite-population, genetic model where each particle interacts with all other particles in the population. In Section 1.3, we construct a stepping stone model on a one-dimensional lattice of equally spaced, discrete sites with Moran-like interactions between particles located at the same site. In Section 1.4, we consider the limiting model that results as we increase the density of the lattice of discrete sites while keeping the *particle* density constant. The particle motions converge to Brownian motions while their limiting interactions are determined by the local times at zero of the distances between particle pairs.

In Chapter 2, we study the stepping stone model in more detail. We begin by introducing an ordering on the particles, in the form of $\mathbb{R}^+$-valued levels. By modifying the model of Section 1.3 so that neutral interactions occur in only one direction determined by this ordering, we construct an ordered model with the same empirical location/type distribution as the original model. The value of this ordered model is that it may be meaningfully extended to infinite densities, as we do in Section 2.3 to construct an infinite-density, neutral stepping stone model.

In Chapter 3, we construct an ordered, infinite-density, Brownian model with local-time interactions and selection. In the neutral case, this ordered model embeds the empirical location/type distribution of the corresponding symmetric Moran model (the limiting model of Section 1.4) for all finite particle densities $K > 0$. The ordered model is conditionally a Poisson point process that carries a location/type measure-valued process ν_t. In fact, ν_t is the limiting measure-valued diffusion of the model of Section 1.4 as the particle density increases to infinity. We prove that ν_t almost surely has vaguely continuous paths and characterize it by means of a collection of martingales whose quadratic variations we calculate.

1.2 Standard Moran Model

The standard Moran model of [9] describes the evolution of a population consisting of two types of particles and having some fixed size n. In this model, birth events occur according to a constant-rate Poisson counting process. When such an event occurs, a random particle is selected to produce an offspring: usually, the offspring has the parent's type, though with some small probability, the offspring mutates to the other type. Then, an existing particle in the population is uniformly randomly chosen to be killed, and the offspring takes its place. Note that births and deaths are balanced.

Many variations of this model are possible. Trivially, it may be generalized to a countable type space with mutation occurring according to transition probabilities between types. It may also be generalized to an uncountable type space via a mutation probability kernel and modified so that particles mutate continuously between reproduction events, with the offspring taking the parent's type at the instant of birth then proceeding to mutate independently of the parent. Additionally, particles may be subject to fertility or viability-based selective forces.

Under certain scalings, the empirical measures of many of these variations can be shown to converge, with increasing population size, to diffusion process limits. As a rule of thumb, if—as the population size n increases—the population-wide rates of neutral and selective reproductive events are kept of order $O(n^2)$ and $O(n)$ respectively, then the type process, expressed as a measure-valued, empirical process assigning mass $\frac{1}{n}$ to each particle, and so taking values in the set $P(E)$ of probability measures on the type space E, will converge to a measure-valued diffusion, under suitable additional conditions. For example, [6] shows convergence for a Moran-like model incorporating selection, recombination, and mutation to a Fleming-Viot limiting process under such a scaling.

In the following section, we will construct a stepping stone model with Moran-like interactions: individuals reproduce at random, replacing other randomly chosen individuals with their offspring. The distinguishing feature of our model will be that the interactions occur only between particles at the same site. In later chapters we will see that the aforementioned scalings can be applied to our model as well: for a particle density of order $O(n)$, if neutral and selective event counts per unit of space per unit of time are of order $O(n^2)$ and $O(n)$ respectively, the limiting empirical type measure will converge to a diffusion process.

1.3 Stepping Stone Model

In this section, we construct a finite-density, stepping stone model with local, Moran-like interactions. The particles "live" on a one-dimensional lattice of equally spaced sites, perform independent simple random walks across the lattice, and interact within each site in a Moran-like fashion: each particle reproduces according to a Poisson counting process, replacing a random particle at the same site.

For each n, we consider a countably infinite population of particles occupying the lattice $n^{-1}\mathbb{Z}$. We assume that the particles move independently each according to a simple random walk at rate θn^2. Note that this scaling is such that the particle motions converge to Brownian motions as $n \to \infty$.

We further assume that each particle $j \in \mathbb{N}$ has a type z_j in a complete, separable metric space E. At every site, each ordered pair of particles (i, j) at that site undergoes reproduction events at rate $\lambda/2 + \sigma(z_i, z_j)$: particle i produces an offspring of type z_i which replaces particle j in the population; equivalently, the type z_j of particle j is changed to the type z_i of particle i. Between reproduction events, the particle types mutate independently.

The reproduction events occurring at rate $\lambda/2$, independent of particle type, will be considered *neutral reproduction events*, while those occurring at rate $\sigma(z_i, z_j)$ will be considered *selective reproduction events*. Here, σ is taken to be a non-negative function bounded above by a constant $\bar{\sigma} > 0$ so that the expression $\sigma(\alpha, \beta) - \sigma(\beta, \alpha)$ represents the selective advantage of type α over type β. In most contexts, we can take $\sigma(\alpha, \beta) \equiv \sigma(\alpha)$ where $\sigma(\alpha)$ represents the absolute selective advantage of type α over some baseline

Formally, for $\boldsymbol{x} \in (n^{-1}\mathbb{Z})^\infty$ and $\boldsymbol{z} \in E^\infty$, the vectors of positions and types of the particles respectively, we define a generator

$$\tilde{A}_n f(\boldsymbol{x}, \boldsymbol{z}) := \sum_j B^\theta_{n,j} f(\boldsymbol{x}, \boldsymbol{z}) + \sum_j B^\mu_j f(\boldsymbol{x}, \boldsymbol{z}) + \sum_{\substack{i \neq j \\ x_i = x_j}} \big(\lambda/2 + \sigma(z_i, z_j)\big) \big(f\big(\boldsymbol{x}, \eta_j(\boldsymbol{z}|z_i)\big) - f(\boldsymbol{x}, \boldsymbol{z})\big) \quad (1.1)$$

where $\eta_j(\boldsymbol{z}|z_0) := (z_1, z_2, \ldots, z_{j-1}, z_0, z_{j+1}, \cdots)$. Here, the migration operator $B^\theta_{n,j}$, which operates on f only as a function of the location x_j of the jth particle, is given by

$$B^\theta_{n,j} f(\boldsymbol{x}, \boldsymbol{z}) := \theta n^2 \left(\frac{f\big(\eta_j(\boldsymbol{x} \mid x_j + 1/n), \boldsymbol{z}\big) + f\big(\eta_j(\boldsymbol{x} \mid x_j - 1/n), \boldsymbol{z}\big)}{2} - f(\boldsymbol{x}, \boldsymbol{z}) \right)$$

Each B^μ_j is some mutation operator B^μ operating on f only as a function of the type z_j of the jth particle. In particular, the mutation is location-independent. Also, note that the last summation in equation (1.1) is taken over all ordered pairs of distinct particles sharing the same location.

We may construct an $(n^{-1}\mathbb{Z})^\infty \times E^\infty$-valued solution $(\mathbf{X}^{(n)}, \tilde{\mathbf{Z}}^{(n)})$ to the martingale problem $\tilde{A}_n$ explicitly on the probability space $(\Omega, \mathfrak{F}, P)$ as follows. As in [2], we assume the existence of a $\mathfrak{B}(E)\times\mathfrak{B}[0,\infty)\times\mathfrak{F}$-measurable mutation mapping

$$Y : E \times [0,\infty) \times \Omega \to E \tag{1.2}$$

such that for all y and ω, we have $Y(y, 0, \omega) = y$ and $Y(y, \cdot, \cdot)$ a Markov process with transition kernel $Q(t, y, dy')$ having sample paths in $D_E[0,\infty)$. We will also assume the following:

Hypothesis 1.3.1. Assume that for each $y \in E$, the process $Y(y, \cdot, \cdot)$ is the unique solution to the martingale problem (B^μ, δ_y), and assume that the generator B^μ is closed under multiplication and has $\|B^\mu g\|_\infty < \infty$ for all bounded $g \in \mathfrak{D}(B^\mu)$.

Let $\mathbf{X}^{(n)}(0)$ and $\tilde{\mathbf{Z}}^{(n)}(0)$ be given. Let $\{W_j^{\mathbb{Z}} : j \in \mathbb{N}\}$ be iid simple, symmetric, rate 1 random walks on $\mathbb{Z}$, let $\{N_{ij}^\lambda, \bar{N}_{ij}^\sigma : i \neq j \in \mathbb{N}\}$ be iid, rate 1 Poisson counting processes, let $\{Y_{jk} : j \in \mathbb{N}, k \in \mathbb{Z}^+\}$ be iid copies of Y, and let $\{\zeta_{jk} : j \in \mathbb{N}, k \in \mathbb{Z}^+\}$ be iid uniform on $[0,1]$ such that $\big(\mathbf{X}^{(n)}(0), \tilde{\mathbf{Z}}^{(n)}(0)\big)$, $\{W_j^{\mathbb{Z}}\}$, $\{N_{ij}^\lambda\}$, $\{\bar{N}_{ij}^\sigma\}$, $\{Y_{jk}\}$, and $\{\zeta_{jk}\}$ are mutually independent. Define the particle location process $\mathbf{X}^{(n)}$ by

$$X_j^{(n)}(t) := X_j^{(n)}(0) + \frac{1}{n} W_j^{\mathbb{Z}}(\theta n^2 t) \tag{1.3}$$

To define the type process $\tilde{\mathbf{Z}}^{(n)}$, we will need to define, for each ordered particle pair (i, j) with $i \neq j$, counting processes for neutral reproduction events (where j copies i's type) and *potential* selective reproduction events (where j *may* copy i's type) as follows. Define the neutral event counting process by

$$\tilde{V}_{ij}^{(n),\lambda}(t) := N_{ij}^\lambda \left(\frac{\lambda}{2} \int_0^t 1_{\{X_i^{(n)}(s) = X_j^{(n)}(s)\}} ds \right) \tag{1.4}$$

and define the potential selective event counting process by

$$\bar{V}_{ij}^{(n),\sigma}(t) := \bar{N}_{ij}^{\sigma}\left(\bar{\sigma}\int_0^t 1_{\{X_i^{(n)}(s)=X_j^{(n)}(s)\}}ds\right) \tag{1.5}$$

where $\bar{\sigma}$ is the upper bound for the selection function σ.

For each j, define the counting process

$$\tilde{V}_j := \sum_{i\neq j}\left(\tilde{V}_{ij}^{(n),\lambda} + \bar{V}_{ij}^{(n),\sigma}\right)$$

This counts all neutral and potential selective reproduction events directly affecting the type of j. For $k \in \mathbb{N}$, let $\tilde{\tau}_{jk}$ be the kth jump time of $\tilde{V}_j$, and define

$$\tilde{\gamma}_{jk} := \sum_{i\neq j} i\Delta\tilde{V}_{ij}^{(n),\lambda}(\tilde{\tau}_{jk}) + \sum_{i\neq j} i\Delta\bar{V}_{ij}^{(n),\sigma}(\tilde{\tau}_{jk})$$
$$\tilde{\iota}_{jk} := \sum_{i\neq j}\Delta\tilde{V}_{ij}^{(n),\lambda}(\tilde{\tau}_{jk})$$

Almost surely, the summations $\tilde{\gamma}_{jk}$ and $\tilde{\iota}_{jk}$ each have exactly one non-zero term. Thus, $\tilde{\gamma}_{jk}$ gives the particle i whose type j either copies in the case of a neutral event (where $\tilde{\iota}_{jk} = 1$) or *may* copy in the case of a potential selective event (where $\tilde{\iota}_{jk} = 0$) at event time $\tilde{\tau}_{jk}$.

Finally, with the conventions $\tilde{\tau}_{j,0} := 0$, $\tilde{\gamma}_{j,0} := j$, $\tilde{\iota}_{j,0} := 1$, and $\tilde{Z}_j^{(n)}(0-) := \tilde{Z}_j^{(n)}(0)$, we may define the type process $\tilde{\mathbf{Z}}^{(n)}$ as the unique solution to the system of equations

$$\tilde{Z}_j^{(n)}(t) = Y_{jk}\left(\phi\left(\tilde{Z}_{\tilde{\gamma}_{jk}}^{(n)}(\tilde{\tau}_{jk}-), \tilde{Z}_j^{(n)}(\tilde{\tau}_{jk}-), \tilde{\iota}_{jk}, \zeta_{jk}\right), t-\tilde{\tau}_{jk}\right), \quad \tilde{\tau}_{jk} \le t < \tilde{\tau}_{j,k+1} \tag{1.6}$$

where ϕ is given by

$$\phi(z_1, z_2, \iota, \zeta) := \begin{cases} z_1, & \zeta \le (\bar{\sigma}^{-1}\sigma(z_1, z_2)) \vee \iota; \\ z_2, & \zeta > (\bar{\sigma}^{-1}\sigma(z_1, z_2)) \vee \iota. \end{cases} \tag{1.7}$$

Remark 1.3.1. Existence and uniqueness of a solution to (1.6) depends on the choice of initial location vector $\mathbf{X}^{(n)}(0)$. In particular, a unique solution will exist if it can be established that, for each fixed time $T > 0$ and particle $j \in \mathbb{N}$, the number of events $\tilde{V}_j(T)$ and the total number of particles that may have "influenced" j's type $\tilde{Z}_j^{(n)}(T)$ at time T through a chain of interaction events are both finite. A rigorous proof would take much the same form as the proof of the existence of the processes $\mathbf{\Phi}$ and $\tilde{\mathbf{U}}$ in Theorem 3.3.1 and Lemma 3.3.2. In particular, under the stationary, Poisson initial distribution described below, this argument applies and a unique solution exists. Note that we refer here only to the uniqueness of the solution to (1.6), not to the uniqueness of the martingale problem for $\tilde{A}_n$.

Remark 1.3.2. In defining the type process $\tilde{\mathbf{Z}}^{(n)}$ above, we first specified neutral $\{\tilde{V}_{ij}^{(n),\lambda}\}$ and potential selective $\{\bar{V}_{ij}^{(n),\sigma}\}$ event counting processes in equations (1.4) and (1.5) respectively. We then defined the processes $\tilde{V}_j$, $\tilde{\tau}_{jk}$, $\tilde{\gamma}_{jk}$, and $\tilde{\iota}_{jk}$ deterministically in terms of $\tilde{V}_{ij}^{(n),\lambda}$ and $\bar{V}_{ij}^{(n),\sigma}$. Finally, $\tilde{\mathbf{Z}}^{(n)}$ was defined as the unique solution to an equation (1.6) involving these derived processes, the initial types $\tilde{\mathbf{Z}}^{(n)}(0)$, the processes Y_{jk}, and the uniform random variables ζ_{jk}.

In later sections, we will need to apply this procedure several times. That is, we will need to define a type process as the unique solution of a system of equations involving initial types, mutation processes, uniform random variables, and several intermediate processes all deterministically dependent on neutral and potential selective event counting processes.

Rather than repeat the derivation each time, we will introduce the following notion. If we write

$$(\boldsymbol{z}, \boldsymbol{\nu}, \boldsymbol{\tau}, \boldsymbol{\gamma}, \boldsymbol{\iota}) := \mathfrak{T}(\boldsymbol{z}_0, \boldsymbol{y}, \boldsymbol{\nu}^\lambda, \bar{\boldsymbol{\nu}}^\sigma, \boldsymbol{\zeta}, \bar{\sigma}^{-1}\sigma)$$

we mean to indicate that we are defining $\boldsymbol{\nu}$ by

$$\nu_j := \sum_{i \neq j} \left(\nu_{ij}^\lambda + \bar{\nu}_{ij}^\sigma \right)$$

defining τ_{jk} to be the ordered jump times of ν_j; defining $\boldsymbol{\gamma}$ and $\boldsymbol{\iota}$ by

$$\gamma_{jk} := \sum_{i\neq j} i\Delta\nu_{ij}^{\lambda}(\tau_{jk}) + \sum_{i\neq j} i\Delta\bar{\nu}_{ij}^{\sigma}(\tau_{jk})$$
$$\iota_{jk} := \sum_{i\neq j} \Delta\nu_{ij}^{\lambda}(\tau_{jk})$$

adopting the conventions $\tau_{j,0} := 0$, $\gamma_{j,0} := j$, $\iota_{j,0} := 1$, and $z_j(0-) := z_{0,j}$; and finally defining $\boldsymbol{z}$ as the unique solution to

$$z_j(t) = y_{jk}\left(\phi\left(z_{\gamma_{jk}}(\tau_{jk}-), z_j(\tau_{jk}-), \iota_{jk}, \zeta_{jk}\right), t-\tau_{jk}\right), \quad \tau_{jk} \leq t < \tau_{j,k+1}$$

In particular, the above formulation could have been written

$$(\tilde{\mathbf{Z}}^{(n)}, \tilde{\mathbf{V}}, \tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\iota}}) := \mathfrak{T}(\tilde{\mathbf{Z}}^{(n)}(0), \mathbf{Y}, \tilde{\mathbf{V}}^{(n),\lambda}, \bar{\mathbf{V}}^{(n),\sigma}, \boldsymbol{\zeta}, \bar{\sigma}^{-1}\sigma) \tag{1.8}$$

The intuitive interpretation of (1.6)—and so (1.8)—is this. Particles mutate independently between interactions. At an interaction time $\tilde{\tau}_{jk}$, the function $\phi(\cdot)$ determines the new type of particle j. In the case of a neutral event, we have $\tilde{\iota}_{jk} = 1$, and j always copies the other particle's type. In the case of a potential selective event, where $\tilde{\iota}_{jk} = 0$, a biased coin is flipped. The bias depends on the selective advantage of the particle $\tilde{\gamma}_{jk}$ over j: if it is large enough, j copies $\tilde{\gamma}_{jk}$'s type; otherwise, j keeps its current type.

To make this intuition more precise, note that we have

$$\bar{V}_{ij}^{(n),\sigma}(t) = \sum_{k=1}^{\infty} 1_{\{\tilde{\tau}_{jk}\leq t\}} 1_{\{\tilde{\gamma}_{jk}=i\}} 1_{\{\tilde{\iota}_{jk}=0\}}$$

so we may define the actual (cf. potential) selective event counting processes as a filtering of $\bar{V}_{ij}^{(n),\sigma}$ given by

$$\tilde{V}_{ij}^{(n),\sigma}(t) := \sum_{k=1}^{\infty} 1_{\{\tilde{\tau}_{jk}\leq t\}} 1_{\{\tilde{\gamma}_{jk}=i\}} 1_{\{\tilde{\iota}_{jk}=0\}} 1_{\left\{\zeta_{jk}\leq\bar{\sigma}^{-1}\sigma\left(\tilde{Z}_i^{(n)}(\tilde{\tau}_{jk}-), \tilde{Z}_j^{(n)}(\tilde{\tau}_{jk}-)\right)\right\}}$$

These processes satisfy

$$\tilde{V}_{ij}^{(n),\sigma}(t) = N_{ij}^{\sigma}\left(\int_0^t \sigma\left(\tilde{Z}_i^{(n)}(s), \tilde{Z}_j^{(n)}(s)\right) 1_{\{X_i^{(n)}(s)=X_j^{(n)}(s)\}} ds\right) \tag{1.9}$$

for iid, rate 1 Poisson counting processes N^{σ}_{ij} independent of other aspects of the model. Furthermore, in the case where there is no mutation, the particle type process $\tilde{\mathbf{Z}}^{(n)}$ then satisfies

$$\tilde{Z}^{(n)}_j(t) = \tilde{Z}^{(n)}_j(0) + \sum_{i \neq j} \int_0^t \left(\tilde{Z}^{(n)}_i(s-) - \tilde{Z}^{(n)}_j(s-) \right) d\left(\tilde{V}^{(n),\lambda}_{ij}(s) + \tilde{V}^{(n),\sigma}_{ij}(s) \right)$$

For convenience, we will write the model constructed above as $\tilde{\Upsilon}_n$. That is, for initial location and type vectors $\mathbf{X}^{(n)}(0) = \mathbf{X}_0$ and $\tilde{\mathbf{Z}}^{(n)}(0) = \mathbf{Z}_0$ respectively, we will define

$$\tilde{\Upsilon}_n(\mathbf{X}_0, \mathbf{Z}_0) := (\mathbf{X}^{(n)}, \tilde{\mathbf{Z}}^{(n)})$$

Implicitly, of course, $\tilde{\Upsilon}_n$ depends on the random variables $W^{\mathbb{Z}}_j$, N^{λ}_{ij}, $\bar{N}^{\sigma}_{ij}$, Y_{jk}, and ζ_{jk} and the parameters θ, λ, and σ defined above.

Let us now describe one special case for the distribution of the initial locations and types $(\mathbf{X}_0, \mathbf{Z}_0)$. Let $\nu^{(n)}_0$ be a measure on location/type space $n^{-1}\mathbb{Z} \times E$ with marginal location measure $\nu^{(n)}_0(\cdot \times E) = \sum_{x \in n^{-1}\mathbb{Z}} \frac{1}{n}\delta_x$, and let $(\mathbf{X}_0, \mathbf{Z}_0)$ be some indexing of the points of a Poisson point process with mean measure $K\nu^{(n)}_0$ for some particle density $K > 0$. The conditions of Remark 1.3.1 hold, so a unique solution to (1.6) exists for this initial distribution. If we define the vector $\mathbf{Y}^{(n)}$ by

$$Y^{(n)}_k(t) := \sum_{j \in \mathbb{N}} 1_{\{X^{(n)}_j(t) = k/n\}} \tag{1.10}$$

which gives, for each $k \in \mathbb{Z}$, the number of particles at site k/n at time t, then the process $\mathbf{Y}^{(n)}$ is stationary, and the $\{Y^{(n)}_k(t)\}_{k \in \mathbb{Z}}$ are iid Poisson with mean K/n for all $t \geq 0$. To see this, we will apply the following Lemma.

Lemma 1.3.2. *Let* E *and* E' *be metric spaces, and let* $h\colon E \to E'$ *be Borel measurable. If* Γ *is a Poisson point process on* E *with* σ*-finite mean measure* ν *on* $\mathfrak{B}(E)$ *such that* $\nu' := \nu \circ h^{-1}$ *is* σ*-finite on* $\mathfrak{B}(E')$*, then for an indexing of the points*

$$\Gamma = \sum_{j \in \mathbb{N}} \delta_{\xi_j}$$

the random measure

$$\Gamma' := \sum_{j\in\mathbb{N}} \delta_{h(\xi_j)}$$

is a Poisson point process on E' *having mean measure* ν'.

Proof. For $A \in \mathfrak{B}(E')$, we have $h^{-1}(A) \in \mathfrak{B}(E)$. Thus, $\Gamma'(A) = \Gamma(h^{-1}(A))$ is Poisson with mean $\nu(h^{-1}(A))$. Also, for disjoint $A, B \in \mathfrak{B}(E')$, we have $h^{-1}(A)$ and $h^{-1}(B)$ disjoint, so $\Gamma'(A) = \Gamma(h^{-1}(A))$ and $\Gamma'(B) = \Gamma(h^{-1}(B))$ are independent. ☐

Corollary 1.3.3. *If* $(\mathbf{X}_0, \mathbf{Z}_0) \sim \mathit{Poisson}(K\nu_0^{(n)})$ *then the per-site particle counts* $\mathbf{Y}^{(n)}(t)$ *are iid Poisson mean* K/n *for all* $t \geq 0$.

Proof. We note that $(X_{0,j}, W_j^{\mathbb{Z}})$ are the points of a Poisson point process on $n^{-1}\mathbb{Z} \times D_{\mathbb{Z}}[0,\infty)$ with mean measure $\nu := (K\sum_{x\in n^{-1}\mathbb{Z}} \frac{1}{n}\delta_x) \times \mathfrak{W}^{\mathbb{Z}}$ where $\mathfrak{W}^{\mathbb{Z}}$ is the law of $W_1^{\mathbb{Z}}$. Since h given by

$$\begin{aligned} h\colon n^{-1}\mathbb{Z} \times D_{\mathbb{Z}}[0,\infty) &\to n^{-1}\mathbb{Z} \\ : (X_{0,j}, W_j^{\mathbb{Z}}) &\mapsto X_j^{(n)}(t) = X_{0,j} + \tfrac{1}{n} W_j^{\mathbb{Z}}(\theta n^2 t) \end{aligned}$$

is measurable and $\nu' := \nu \circ h^{-1}$ is σ-finite, we have $\Gamma' := \sum_j \delta_{X_j^{(n)}(t)}$ a Poisson point process on $n^{-1}\mathbb{Z}$ with mean measure ν' by Lemma 1.3.2.

In particular, we have $Y_k^{(n)}(t) = \Gamma'(\{k/n\})$, the Γ'-measures of disjoint sets. To prove the result, it remains only to show that $\nu'(\{k/n\}) = K/n$ for all $k \in \mathbb{Z}$. This may be calculated explicitly, or we may observe that it follows from the initial distribution and the symmetry of the system. ☐

Thus, for model $\tilde{\Upsilon}_n$ with $(\mathbf{X}_0, \mathbf{Z}_0)$ distributed $\mathit{Poisson}(K\nu_0^{(n)})$ which we may write

$$(\mathbf{X}^{(n)}, \tilde{\mathbf{Z}}^{(n)}) = \tilde{\Upsilon}_n\big(\mathit{Poisson}(K\nu_0^{(n)})\big)$$

the interpretation is as follows. We begin with the lattice $n^{-1}\mathbb{Z}$ populated by iid, mean K/n Poisson particles per site. These particles migrate according to iid, rate θn^2 simple random walks $W_j^{\mathbb{Z}}(\theta n^2 t)/n$ so that the initial location distribution is stationary. The processes $\tilde{V}_{ij}^{(n),\lambda}$ count neutral reproduction events whereby particle i produces an offspring replacing particle j (or, equivalently, particle j "looks at" particle i and copies its type); and the processes $\tilde{V}_{ij}^{(n),\sigma}$ count the selective reproduction events analogously. Observe that $\tilde{V}_{ij}^{(n),\lambda}$ and $\tilde{V}_{ij}^{(n),\sigma}$ only increase while particle i and particle j share the same site. Between reproduction events, the particles mutate independently according to the generator B^{μ}.

1.4 Brownian Model

In the previous section we observed that the scaling was such that individual particle motions converged, as $n \to \infty$, to Brownian motions. In this section, we will fix a particle density $K > 0$ and let $n \to \infty$. Under a suitable scaling of the interaction parameters λ and σ, the reproduction event counting processes will converge to non-trivial limits, and the result will be a K-density, interacting particle model freed of the discrete sites of the previous section. Moreover, we will be able to characterize, in the limit, the reproduction events between an ordered pair of particles as resulting from Poisson counting processes driven by the local time at zero of their distance from each other.

We define the local time in the sense of [10]. Let sgn be the left-continuous sign function given by

$$\operatorname{sgn}(x) := \begin{cases} 1, & x > 0; \\ -1, & x \leq 0. \end{cases}$$

Let X be a semimartingale. By [10] Theorem IV.47, there is a unique adapted, càdlàg, increasing process A^a such that

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_{s-} - a)\, dX_s + A_t^a \tag{1.11}$$

Definition 1.4.1. Let X be a semimartingale, and let A^a be given by (1.11). The *local time* $L^a(X)$ *at* a *of* X is the continuous part of A^a given by

$$L_t^a(X) := A_t^a - \sum_{0<s\leq t} \left(|X_s - a| - |X_{s-} - a| - \operatorname{sgn}(X_{s-} - a)\Delta X_s\right)$$

Now, let us define a sequence of models $\left(\mathbf{X}^{(n)}, \tilde{\mathbf{Z}}^{(n)}\right)$ as follows. Let ν_0 be a measure on location/type space $\mathbb{R} \times E$ with marginal measure $\nu_0(\cdot \times E)$

the Lebesgue measure $\ell_{\mathbb{R}}$ on $\mathbb{R}$. Fix a particle density $K > 0$, and let the vector $(\mathbf{X}_0, \mathbf{Z}_0)$ be some indexing of the points of a Poisson point process with mean measure $K\nu_0$. For each $n \in \mathbb{N}$, if we take $X_j^{(n)}(0) = \lfloor nX_{0,j} \rfloor / n$, note that (by Lemma 1.3.2, for instance) we have $(\mathbf{X}^{(n)}(0), \mathbf{Z}_0)$ an indexing of the points of a Poisson point process with mean measure $K\nu_0^{(n)}$ where $\nu_0^{(n)}$ is given by

$$\nu_0^{(n)}(\{k/n\} \times B) = \nu_0\left(\left[\frac{k}{n}, \frac{k+1}{n}\right) \times B\right) \tag{1.12}$$

and has marginal location measure $\nu_0^{(n)}(\cdot \times E) = \sum_{x \in n^{-1}\mathbb{Z}} \frac{1}{n}\delta_x$. Consider the model

$$\left(\mathbf{X}^{(n)}, \tilde{\mathbf{Z}}^{(n)}\right) = \tilde{\Upsilon}_n\left(\mathbf{X}^{(n)}(0), \mathbf{Z}_0\right) \tag{1.13}$$

Note that Corollary 1.3.3 applies, and the $Y_k^{(n)}(t)$ defined by (1.10) are iid Poisson with mean K/n for all $t \geq 0$.

As $n \to \infty$, the initial particle locations $\mathbf{X}^{(n)}(0)$ converge to $\mathbf{X}_0$. Moreover, the particle location processes $X_j^{(n)}$, conditioned on $\mathbf{X}_0$, converge to independent Brownian motions. That is, we have

$$\mathbf{X}^{(n)} \Rightarrow \mathbf{X}$$

where the $\mathbf{X}$ are given by

$$X_j := X_{0,j} + \sqrt{\theta}W_j \tag{1.14}$$

for $\{W_j : j \in \mathbb{N}\}$ iid Brownian motions independent of $\mathbf{X}_0$. Define the distance $X_{ij}^{(n)}$ between particles i and j by

$$X_{ij}^{(n)}(t) := X_i^{(n)}(t) - X_j^{(n)}(t) \tag{1.15}$$

and the limiting distance X_{ij} by

$$X_{ij}(t) := X_i(t) - X_j(t)$$

so that

$$\left(X_{ij}^{(n)}\right)_{i \neq j} \Rightarrow (X_{ij})_{i \neq j} \tag{1.16}$$

The type process $\tilde{Z}^{(n)}$ is driven by the $\tilde{V}_{ij}^{(n),\lambda}$ and $\bar{V}_{ij}^{(n),\sigma}$ counting processes whose rates are proportional to the integral

$$\int_0^t 1_{\{X_i^{(n)}(s)=X_j^{(n)}(s)\}} ds = \int_0^t 1_{\{X_{ij}^{(n)}(s)=0\}} ds$$

By means of the following theorem, we will characterize the limiting distribution of these integrals in terms of the X_{ij} defined above.

First, we will need the following form of the weak law of large numbers.

Lemma 1.4.2. *Let $\xi_{n,k}$ be identically distributed with finite mean μ such that $\{\xi_{n,k}\}_k$ are independent for each n. If $N(n) \xrightarrow{p} \infty$, then*

$$\frac{1}{N(n)} \sum_{k=1}^{N(n)} \xi_{n,k} \xrightarrow{p} \mu$$

Proof. Define

$$h(\epsilon, K_0) := P\left\{ \sup_{K \geq K_0} \left| \frac{1}{K} \sum_{k=1}^{K} \xi_{1,k} - \mu \right| > \epsilon \right\}$$

By the strong law of large numbers

$$\frac{1}{K} \sum_{k=1}^{K} \xi_{1,k} \xrightarrow{a.s.} \mu$$

so we have $h(\epsilon, K_0) \to_{K_0 \to \infty} 0$ for each $\epsilon > 0$. However,

$$P\left\{ \left| \frac{1}{N(n)} \sum_{k=1}^{N(n)} \xi_{n,k} - \mu \right| > \epsilon \right\} \leq h(\epsilon, K_0) + P(N(n) < K_0) \to 0$$

as $K_0 \to \infty$ for all $\epsilon > 0$. □

We will also need this technical lemma. Its importance is the relatively weak form of independence between $N(n)$ and the ξ_i.

Lemma 1.4.3. *Let $\{\xi_i\}$ be identically distributed with $\mathrm{E}|\xi_1| < \infty$. For any positive, integer-valued random sequence $N(n)$ such that $N(n)$ and $\xi_{N(n)}$ are independent for each n, we have $\xi_{N(n)}/n \xrightarrow{a.s.} 0$.*

Proof. Fix any $\epsilon > 0$, and define $A_n := \{\omega : |\xi_{N(n)}| > n\epsilon\}$.

Note that

$$\sum_{n=1}^{\infty} \mathrm{P}\, A_n = \sum_{n=1}^{\infty} \mathrm{P}\left(|\xi_1| > n\epsilon\right) \leq \int_0^{\infty} \mathrm{P}\left(\frac{|\xi_1|}{\epsilon} > t\right) dt = \frac{\mathrm{E}\,|\xi_1|}{\epsilon} < \infty$$

Therefore, by the Borel-Cantelli Lemma, $\mathrm{P}(A_n \text{ i.o.}) = 0$.

However, since this holds for all $\epsilon > 0$, we have $\xi_{N(n)}/n \xrightarrow{a.s.} 0$. □

Finally, we turn to our theorem, which characterizes the limiting duration time at zero of a collection of sequences of random walks in terms of the local time at zero of their limiting processes.

Theorem 1.4.4. *Let* $\{V^k\}$ *for* $k = 1, \ldots, K$ *be a collection of (not necessarily independent)* $\mathbb{Z}$*-valued, continuous-time Markov jump processes started at* $V^k(0) = 0$ *with transition intensities given by*

$$q_{ij} = \begin{cases} \theta, & j = i \pm 1; \\ 0, & \textit{otherwise.} \end{cases}$$

Define $W_n^k(t) := w_n^k - V^k(n^2 t)/n$ *such that* $n w_n^k \in \mathbb{Z}$ *and* $\lim_n w_n^k$ *exists for each* k*. If*

$$(W_n^1, \ldots, W_n^K) \Rightarrow (W_0^1, \ldots, W_0^K) \tag{1.17}$$

then

$$\left(W_n^k,\ n\int_0^{\cdot} 1_{\{W_n^k(s-)=0\}} 2\theta ds\right)_{k=1,\ldots,K} \Rightarrow (W_0^k, L^k)_{k=1,\ldots,K} \tag{1.18}$$

where L^k *are the local times at zero of* W_0^k*.*

Proof. Let sgn be the left-continuous sign function. By Theorem 1 of [11], we have

$$\left(W_n^k,\ \left|W_n^k(\cdot)\right| - \left|W_n^k(0)\right| - \int_0^{\cdot} \operatorname{sgn} W_n^k(s-) dW_n^k(s)\right)_k$$
$$\Rightarrow \left(W_0^k,\ \left|W_0^k(\cdot)\right| - \left|W_0^k(0)\right| - \int_0^{\cdot} \operatorname{sgn} W_0^k(s-) dW_0^k(s)\right)_k \tag{1.19}$$

However, by the Meyer-Tanaka Formula [10, p. 169], the right hand side is $(W_0^k, L^k)_k$ where L^k are the local times at zero of the W_0^k.

Define the ordered exit times of V^k from nw_n^k (and so of W_n^k from 0) by

$$
\begin{aligned}
T_0^{k,n} &:= 0 \\
T_m^{k,n} &:= \inf\{t > T_{m-1}^{k,n} : V^k(t-) = nw_n^k, V^k(t) \neq nw_n^k\}
\end{aligned}
$$

Let $\tau_m^{k,n}$ be the duration times at nw_n^k given by

$$\tau_m^{k,n} := T_m^{k,n} - \sup\{t < T_m^{k,n} : V^k(t) \neq nw_n^k\} \vee 0$$

Define $N^{k,n}(t) := \sup\{m : T_m^{k,n} \leq t\}$, the counting process of V^k's exits from nw_n^k.

Note, by the recurrence of V^k, we have $N^{k,n}(t) \xrightarrow{a.s.}_{t\to\infty} \infty$ for any fixed n. To establish that $N^{k,n}(n^2t) \xrightarrow{p}_{n\to\infty} \infty$ for every $t \in \mathbb{R}^+$, we need only show that $P(N^{k,n}(n^2t) = 0) \longrightarrow_{n\to\infty} 0$. However, the nw_n^k are eventually bounded by $n(w^k + 1)$, so it follows that

$$P(N^{k,n}(n^2t) = 0) \leq P(V^k(s) \leq n(w^k + 1)\ \forall s \leq n^2t) \to 0$$

by the properties of random walks.

For each fixed n and k, note that $V^k(T_m^{k,n})$ are iid for all m with distribution given by

$$P\big(V^k(T_m^{k,n}) = nw_n^k + 1\big) = P\big(V^k(T_m^{k,n}) = nw_n^k - 1\big) = \frac{1}{2}$$

and the $\tau_m^{k,n}$ are iid exponential for all m with $\mathrm{E}\,\tau_m^{k,n} = (2\theta)^{-1}$.

For all $t > 0$, we see that

$$
\begin{aligned}
&\big|W_n^k(t)\big| - \big|W_n^k(0)\big| - \int_0^t \operatorname{sgn} W_n^k(s-)\, dW_n^k(s) \\
&\qquad = 2 \sum_{m=1}^{N^{k,n}(n^2t)} \frac{1}{n} \left(V^k(T_m^{k,n}) - nw_n^k\right)^+ \\
&\qquad = 2\frac{N^{k,n}(n^2t)}{n} \frac{1}{N^{k,n}(n^2t)} \sum_{m=1}^{N^{k,n}(n^2t)} \left(V^k(T_m^{k,n}) - nw_n^k\right)^+
\end{aligned}
$$

By Lemma 1.4.2, we have

$$\frac{1}{N^{k,n}(n^2t)} \sum_{m=1}^{N^{k,n}(n^2t)} \left(V^k(T^{k,n}_m) - nw^k_n\right)^+ \xrightarrow{p} \frac{1}{2}$$

as $n \to \infty$ for all $t \in \mathbb{R}^+$.

Therefore, by equation (1.19), we have

$$\left(W^k_n, \frac{N^{k,n}(n^2\cdot)}{n}\right)_k \Rightarrow \left(W^k_0, L^k\right)_k \tag{1.20}$$

However,

$$\int_0^t 1_{\{W^k_n(s-)=0\}} ds = \sum_{m=1}^{N^{k,n}(n^2t)} \frac{\tau^{k,n}_m}{n^2} + \frac{1}{n^2}\left(t - T^{k,n}_{N^{k,n}(n^2t)+1} + \tau^{k,n}_{N^{k,n}(n^2t)+1}\right) \vee 0$$

Thus,

$$n\int_0^t 1_{\{W^k_n(s-)=0\}} 2\theta ds = 2\theta \frac{N^{k,n}(n^2t)}{n} \frac{1}{N^{k,n}(n^2t)} \sum_{m=1}^{N^{k,n}(n^2t)} \tau^{k,n}_m + 2\frac{\theta}{n}\left(t - T^{k,n}_{N^{k,n}(n^2t)+1} + \tau^{k,n}_{N^{k,n}(n^2t)+1}\right) \vee 0$$

Applying Lemma 1.4.2, we have

$$\frac{1}{N^{k,n}(n^2t)} \sum_{m=1}^{N^{k,n}(n^2t)} \tau^{k,n}_m \xrightarrow{p} \frac{1}{2\theta}$$

as $n \to \infty$ for all $t \in \mathbb{R}^+$. By Lemma 1.4.3, we have

$$0 \le \frac{\theta}{n}\left(t - T^{k,n}_{N^{k,n}(n^2t)+1} + \tau^{k,n}_{N^{k,n}(n^2t)+1}\right) \vee 0 \le \theta \frac{\tau^{k,n}_{N^{k,n}(n^2t)+1}}{n} \xrightarrow{a.s.} 0$$

as $\tau^{k,n}_{N^{k,n}(n^2t)+1}$ is independent of $N^{k,n}(n^2t)$ for all n.

Therefore, it follows from equation (1.20) that equation (1.18) holds. □

Proposition 1.4.5. *We have*

$$\left(X^{(n)}_i, X^{(n)}_j, n\int_0^\cdot 1_{\{X^{(n)}_i(s)=X^{(n)}_j(s)\}} 2\theta ds\right)_{i\neq j} \Rightarrow \left(X_i, X_j, L_{ij}\right)_{i\neq j} \tag{1.21}$$

where the L_{ij} *are local times at zero of the processes* $X_{ij} := X_i - X_j$.

Proof. We have

$$X_{ij}^{(n)}(t) = \left(X_i^{(n)}(0) - X_j^{(n)}(0)\right) + \frac{1}{n}\left(W_i^{\mathbb{Z}}(\theta n^2 t) - W_j^{\mathbb{Z}}(\theta n^2 t)\right)$$

so conditioned on $\mathbf{X}_0$, the $X_{ij}^{(n)}$ satisfy Theorem 1.4.4 with starting points

$$w_{ij}^{(n)} = X_i^{(n)}(0) - X_j^{(n)}(0) \xrightarrow{a.s.} X_{0,i} - X_{0,j}$$

The characterization of the limiting process (1.16) gives the convergence (1.21) conditioned on $\mathbf{X}_0$. It follows from the independence of $\mathbf{X}_0$ and $\{W_j\}$ that the convergence is unconditional as well. □

In particular, Proposition 1.4.5 allows us to characterize the limiting interactions between pairs of particles. The convergence (1.21) implies that, if λ and $\bar{\sigma}$ are held constant while $n \to \infty$, then

$$\tilde{V}_{ij}^{(n),\lambda}(t) + \bar{V}_{ij}^{(n),\sigma}(t) \xrightarrow{p} 0$$

However, (1.21) also suggests the correct parameter scaling to obtain nontrivial limits. Let $\lambda_0 > 0$ and $\sigma_0(\cdot,\cdot) \geq 0$ be given, and let $\sigma_0 \leq \bar{\sigma}_0$ for some bound $\bar{\sigma}_0$. Take $\lambda = \lambda_0 n$, $\sigma = \sigma_0 n$, and $\bar{\sigma} = \bar{\sigma}_0 n$ so that $\sigma(\cdot,\cdot) \leq \bar{\sigma}$. With these parameter choices, we have the following characterization of the limiting event processes.

Proposition 1.4.6. *Let* $\tilde{V}_{ij}^{(n),\lambda}$ *be given by* (1.4) *with* $\lambda = \lambda_0 n$*, let* $\bar{V}_{ij}^{(n),\sigma}$ *be given by* (1.5) *with* $\bar{\sigma} = \bar{\sigma}_0 n$*, and let* L_{ij} *be given as in Proposition 1.4.5. Then, we have*

$$\left(\tilde{V}_{ij}^{(n),\lambda}, \bar{V}_{ij}^{(n),\sigma}\right)_{i\neq j} \Rightarrow \left(N_{ij}^{\lambda}\left(\frac{\lambda_0}{4\theta}L_{ij}\right), \bar{N}_{ij}^{\sigma}\left(\frac{\bar{\sigma}_0}{2\theta}L_{ij}\right)\right)_{i\neq j}$$

Proof. Since we have

$$\left(n\int_0^{\cdot} 1_{\{X_i^{(n)}(s)=X_j^{(n)}(s)\}} 2\theta\, ds\right)_{i\neq j} \Rightarrow (L_{ij})_{i\neq j}$$

the result follows from the independence of N_{ij}^{λ} and $\bar{N}_{ij}^{\sigma}$ from the $X_j^{(n)}$ and L_{ij}. □

Armed with the result of Proposition 1.4.6, we may define limiting neutral event counting processes by

$$\tilde{V}^{\lambda}_{ij} := N^{\lambda}_{ij}\left(\frac{\lambda_0}{4\theta}L_{ij}\right)$$

and potential selective event counting processes by

$$\bar{V}^{\sigma}_{ij} := \bar{N}^{\sigma}_{ij}\left(\frac{\bar{\sigma}_0}{2\theta}L_{ij}\right)$$

Then, in the notation of Remark 1.3.2, we may define a type process $\tilde{\mathbf{Z}}$ (cf. definition (1.8)) by

$$(\tilde{\mathbf{Z}}, \tilde{\mathbf{V}}, \tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\iota}}) := \mathfrak{T}(\mathbf{Z}_0, \mathbf{Y}, \tilde{\mathbf{V}}^{\lambda}, \bar{\mathbf{V}}^{\sigma}, \boldsymbol{\zeta}, \bar{\sigma}_0^{-1}\sigma_0) \tag{1.22}$$

Existence and uniqueness of such a $\tilde{\mathbf{Z}}$ is subject to the same considerations as the existence of a solution $\tilde{\mathbf{Z}}^{(n)}$ to (1.6) addressed in Remark 1.3.1. Since we can establish that the $\tilde{V}_j$ have no explosions and that the collection of all particles whose type may have influenced particle j up to time T through chains of interactions is finite, the existence and uniqueness follow.

Defining the actual selective event counting processes by

$$\tilde{V}^{\sigma}_{ij}(t) := \sum_{k=1}^{\infty} 1_{\{\tilde{\tau}_{jk}\le t\}} 1_{\{\tilde{\gamma}_{jk}=i\}} 1_{\{\tilde{\iota}_{jk}=0\}} 1_{\left\{\zeta_{jk}\le \bar{\sigma}_0^{-1}\sigma_0\left(\tilde{Z}_i(\tilde{\tau}_{jk}-),\tilde{Z}_j(\tilde{\tau}_{jk}-)\right)\right\}}$$

we see that these processes satisfy

$$\tilde{V}^{\sigma}_{ij}(t) = N^{\sigma}_{ij}\left(\int_0^t \sigma_0\left(\tilde{Z}_i(s), \tilde{Z}_j(s)\right)(2\theta)^{-1}dL_{ij}(s)\right)$$

for iid rate 1 Poisson counting processes N^{σ}_{ij}. In the case without mutation, the type process $\tilde{\mathbf{Z}}$ satisfies

$$\tilde{Z}_j(t) = \tilde{Z}_j(0) + \sum_{i\ne j}\int_0^t \left(\tilde{Z}_i(s-) - \tilde{Z}_j(s-)\right) d\left(\tilde{V}^{\lambda}_{ij}(s) + \tilde{V}^{\sigma}_{ij}(s)\right)$$

We will write the above model, with locations given by (1.14) and types given by (1.22), as $\tilde{\Upsilon}_0$ so that

$$\tilde{\Upsilon}_0(\mathbf{X}_0, \mathbf{Z}_0) := (\mathbf{X}, \tilde{\mathbf{Z}})$$

We will also define the empirical location/type process $\tilde{\xi}^K$ by

$$\tilde{\xi}^K_t := \sum_j \delta_{(X_j(t),\tilde{Z}_j(t))} \tag{1.23}$$

It should be noted that we have not actually proved a limit theorem for the type process here. We have merely shown that the location and interaction processes of model $\tilde{\Upsilon}_n$ converge to the corresponding processes of model $\tilde{\Upsilon}_0$.

The interpretation of the model

$$\tilde{\Upsilon}_0(\mathit{Poisson}(K\nu_0))$$

is as follows. We begin with the real line $\mathbb{R}$ uniformly populated by a K-density, Poisson collection of particles. The particles migrate as independent Brownian motions so that the location distribution is stationary. The neutral $\tilde{V}^\lambda_{ij}$ and selective $\tilde{V}^\sigma_{ij}$ events (where j copies i's type) are Poisson counting processes driven by the local time at zero of the distance between i and j: in particular, these interactions occur only when i and j are at the same location. Between interactions, the particle types mutate independently.

Chapter 2

Infinite-Density Stepping Stone Model

"Good order is the foundation of all things."

— *Edmund Burke (1729–97)*

2.1 Ordered Particle Construction

In Section 1.3, we considered a stepping stone model $\tilde{\Upsilon}_n$ with finite particle density K. In this chapter, we will consider the effect of increasing the particle density to infinity.

Recall from the discussion in Section 1.2 that, for many Moran model variations without spatial structure, as the population size increases with suitable scalings of the interaction parameters, the empirical type process converges to a $P(E)$-valued limiting process. It would seem that the diffusion limit is somehow modeling an infinite collection of particles interacting in much the same fashion as their finite-population cousins, albeit at much faster rates. But, as [3] points out,

> while it might be convenient in applications to think of the measure-valued diffusion as describing the evolution of a hypothetically infinite population, it is difficult to make this precise.

The point of [3], and [2] before it, is to provide infinite-particle constructions that *do* make this idea precise, justifying the intuition.

The technique used in those papers ultimately consists of dynamically reordering the particles of each finite Moran model of size n (without changing its empirical distribution) in such a way that the distributions of the new ordered models are consistent. That is, the ordered Moran model of size n has the same distribution as the first n particles of the ordered Moran model of size $n+1$. This implies the existence of an infinite ordered particle system with a natural embedding of each finite Moran model of size n: the initial n particles from the infinite system form an ordered Moran model, and a random permutation of those n particles forms the usual symmetric Moran model. The infinite ordered model, it should be noted, carries the limiting measure-valued diffusion process as its limiting empirical measure.

We wish to produce a similar construction for our spatially structured models. The key idea is the modification of this technique used in [1] and [8]. In short, we assign an iid, continuous random variable—a *level*—to each particle. The usual linear ordering $\leq$ of the levels gives an almost surely linear ordering of the particles since, almost surely, no two particles share the same level. The levels are assigned independently of the particles' initial locations and types. Unlike the locations and types, the levels remain constant over time.

In the remainder of this section, we will construct an ordered analogue Υ_n of the symmetric model $\tilde{\Upsilon}_n$ with finite density K. The primary importance of Υ_n lies in its relation to the original model $\tilde{\Upsilon}_n$, so in Section 2.2, we will establish that, ignoring the levels, the two models share the same empirical location/type distribution. We will do this by means of a coupling argument using the filtered martingale problem machinery of [7]. In Section 2.3, we will construct an infinite-density ordered model without selection.

Fix some $n \in \mathbb{N}$ and $K > 0$, and let $\mathbf{X}^{(n)}(0)$ and $\mathbf{Z}^{(n)}(0)$ be given. As in the previous chapter, take $\{W_j^{\mathbb{Z}}\}$ to be iid simple, symmetric, rate 1 random walks, $\{N_{ij}^{\lambda}\}$ and $\{\bar{N}_{ij}^{\sigma}\}$ to be iid rate 1 Poisson counting processes, $\{Y_{jk}\}$ to be iid copies of the mutation process Y, and $\{\zeta_{jk}\}$ to be iid uniform on $[0,1]$. Let $\{U_j : j \in \mathbb{N}\}$ be iid uniform on $[0, K]$ such that $\big(\mathbf{X}^{(n)}(0), \mathbf{Z}^{(n)}(0)\big)$, $\{W_j^{\mathbb{Z}}\}$,

$\{N^{\lambda}_{ij}\}$, $\{\bar{N}^{\sigma}_{ij}\}$, $\{Y_{jk}\}$, $\{\zeta_{jk}\}$, and $\{U_j\}$ are mutually independent.

Consider the location/type/level process $(\mathbf{X}^{(n)}, \mathbf{Z}^{(n)}, \mathbf{U})$ where the location process $\mathbf{X}^{(n)}$ is defined as in equation (1.3) and the level "process" $\mathbf{U}$ is constant $\mathbf{U} \equiv (U_j)$ over time.

We will construct the type process $\mathbf{Z}^{(n)}$ in the much the same manner as we constructed $\tilde{\mathbf{Z}}^{(n)}$ in Section 1.3. Define neutral and potential selective event counting processes by

$$V^{(n),\lambda}_{ij}(t) := 1_{\{U_i<U_j\}} N^{\lambda}_{ij}\left(\lambda \int_0^t 1_{\{X^{(n)}_i(s)=X^{(n)}_j(s)\}} ds\right) \tag{2.1}$$

$$\bar{V}^{(n),\sigma}_{ij}(t) := \bar{N}^{\sigma}_{ij}\left(\bar{\sigma} \int_0^t 1_{\{X^{(n)}_i(s)=X^{(n)}_j(s)\}} ds\right) \tag{2.2}$$

and, in the notation of Remark 1.3.2, define the type process $\mathbf{Z}^{(n)}$ by

$$(\mathbf{Z}^{(n)}, \mathbf{V}, \boldsymbol{\tau}, \boldsymbol{\gamma}, \boldsymbol{\iota}) := \mathfrak{T}(\mathbf{Z}^{(n)}(0), \mathbf{Y}, \mathbf{V}^{(n),\lambda}, \bar{\mathbf{V}}^{(n),\sigma}, \boldsymbol{\zeta}, \bar{\sigma}^{-1}\sigma)$$

Note that we will have existence and uniqueness of such a Z under the conditions of Remark 1.3.1.

Observe that, if we define the actual selective event counting processes by

$$V^{(n),\sigma}_{ij}(t) := \sum_{k=1}^{\infty} 1_{\{\tau_{jk}\le t\}} 1_{\{\gamma_{jk}=i\}} 1_{\{\iota_{jk}=0\}} 1_{\left\{\zeta_{jk}\le \bar{\sigma}^{-1}\sigma\left(Z^{(n)}_i(\tau_{jk}-), Z^{(n)}_j(\tau_{jk}-)\right)\right\}}$$

then these processes satisfy

$$V^{(n),\sigma}_{ij}(t) = N^{\sigma}_{ij}\left(\int_0^t \sigma\left(Z^{(n)}_i(s), Z^{(n)}_j(s)\right) 1_{\{X^{(n)}_i(s)=X^{(n)}_j(s)\}} ds\right) \tag{2.3}$$

for iid, rate 1 Poisson counting processes N^{σ}_{ij} independent of other aspects of the model. In the case without mutation, the type process $\mathbf{Z}^{(n)}$ satisfies

$$Z^{(n)}_j(t) = Z^{(n)}_j(0) + \sum_{i\neq j} \int_0^t \left(Z^{(n)}_i(s-) - Z^{(n)}_j(s-)\right) d\left(V^{(n),\lambda}_{ij}(s) + V^{(n),\sigma}_{ij}(s)\right)$$

Comparing equations (1.4) and (1.9) with (2.1) and (2.3) respectively, we see that, while selective reproduction events occur analogously to those in the model of Section 1.3, neutral reproduction events between a pair of particles occur at twice the rate but *only in one direction*: the particle with the lower level replaces the particle with the higher level or, equivalently, the particle with the higher level "looks down" to copy the lower-level particle's type.

The generator for $(\mathbf{X}^{(n)}, \mathbf{Z}^{(n)}, \mathbf{U})$ is given by

$$\begin{aligned} A_n f(\mathbf{x}, \mathbf{z}, \mathbf{u}) := \sum_j B^\theta_{n,j} f(\mathbf{x}, \mathbf{z}, \mathbf{u}) + \sum_j B^\mu_j f(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ + \sum_{\substack{i \neq j \\ x_i = x_j}} \left(\lambda 1_{\{u_i < u_j\}} + \sigma(z_i, z_j)\right) \\ \left(f\left(\mathbf{x}, \eta_j(\mathbf{z}|z_i), \mathbf{u}\right) - f(\mathbf{x}, \mathbf{z}, \mathbf{u})\right) \end{aligned} \tag{2.4}$$

We will write this model as Υ_n so that, for initial location and type vectors $\mathbf{X}_0$ and $\mathbf{Z}_0$ respectively, we have

$$\Upsilon_n(\mathbf{X}_0, \mathbf{Z}_0, K) := (\mathbf{X}^{(n)}, \mathbf{Z}^{(n)}, \mathbf{U})$$

Note that we have explicitly made K, which determines the distribution of $\mathbf{U}$, a parameter of the model.

Consider the special case for initial location/type distribution described in Section 1.3. If we take $\nu_0^{(n)}$ to be a measure on location/type space $n^{-1}\mathbb{Z} \times E$ with marginal location measure $\nu_0^{(n)}(\cdot \times E) = \sum_{x \in n^{-1}\mathbb{Z}} \frac{1}{n}\delta_x$ and let $(\mathbf{X}_0, \mathbf{Z}_0)$ be some indexing of the points of a Poisson point process with mean measure $K\nu_0^{(n)}$ (so that K may once again be interpreted as the particle density), then $(\mathbf{X}_0, \mathbf{Z}_0, \mathbf{U})$ is an indexing of the points of a Poisson point process with mean measure $\nu_0^{(n)} \times \ell_{[0,K]}$. Moreover, by an argument similar to the proof of Corollary 1.3.3, the location/level process $(\mathbf{X}^{(n)}(t), \mathbf{U})$ is stationary in t, having distribution *Poisson* $\left(\sum_{x \in n^{-1}\mathbb{Z}} \frac{1}{n}\delta_x \times \ell_{[0,K]}\right)$.

2.2 Coupling to the Stepping Stone Moran Model

In this section we will establish that, for a fixed density $K > 0$, the models $\tilde{\Upsilon}_n$ and Υ_n have the same empirical particle location/type distribution. More precisely, we will prove the following.

Proposition 2.2.1. *Fix $K > 0$, and let $\mathbf{X}^{(n)}$, $\tilde{\mathbf{Z}}^{(n)}$, $\mathbf{Z}^{(n)}$, and $\mathbf{U}$ be defined as in Sections 1.3 and 2.1 so that*

$$(\mathbf{X}^{(n)}, \tilde{\mathbf{Z}}^{(n)}) = \tilde{\Upsilon}_n(\mathbf{X}_0, \mathbf{Z}_0)$$

and

$$(\mathbf{X}^{(n)}, \mathbf{Z}^{(n)}, \mathbf{U}) = \Upsilon_n(\mathbf{X}_0, \mathbf{Z}_0, K)$$

Assume the $\mathbf{X}_0$ are such that the conditions of Remark 1.3.1 hold.

If we define empirical location/type processes

$$\tilde{\Gamma}_t = \sum_j \delta_{(X_j^{(n)}(t), \tilde{Z}_j^{(n)}(t))}$$
$$\Gamma_t = \sum_j \delta_{(X_j^{(n)}(t), Z_j^{(n)}(t))}$$

then we have $\tilde{\Gamma} =^d \Gamma$, as measure-valued processes.

To prove this, we will require the following theorem.

Theorem 2.2.2. *Let E, E_1, and E_2 be separable metric spaces, and let γ_1 and γ_2 be maps*

$$E \xrightarrow{\gamma_1} E_1 \times E_2 \xrightarrow{\gamma_2} E_1$$

where γ_1 is continuous and $\gamma_2(x, u) = x$ is the projection map.

Let $A \subseteq B(E) \times B(E)$ be given. Suppose

- *There exists A_1 with $\mathfrak{D}(A_1) = \{h : h \circ \gamma_1 \in \mathfrak{D}(A)\}$ such that*

  $$A(h \circ \gamma_1) = (A_1 h) \circ \gamma_1$$

  for all $h \in \mathfrak{D}(A_1)$;

- *There exists A_2 such that*

  $$\int A_1 h(x, u)\pi(du) = A_2 \int h(x, u)\pi(du)$$

  for some $\pi \in P(E_2)$ and all $h \in \mathfrak{D}(A_1)$.

Fix $\nu_1 \in P(E_1)$*. Let* Y *be a càdlàg solution to the martingale problem for* (A_2, ν_1)*. Let* Z *be a càdlàg solution to the martingale problem for* (A, μ) *for some* $\mu \in P(E)$ *such that* $\nu_1 \times \pi = \mu \circ \gamma_1^{-1}$*.*

If A_1 *satisfies the conditions of [7] Theorem 2.6 and uniqueness holds for the martingale problem for* $(A_1, \nu_1 \times \pi)$*, then* $Y =^d \gamma_2 \circ \gamma_1 \circ Z$*.*

Proof. Define the $P(E_1 \times E_2)$-valued process $\tilde{\pi}$ by $\tilde{\pi}_t(A, B) := 1_{\{Y(t) \in A\}} \pi(B)$ for all $A \in \mathfrak{B}(E_1)$ and $B \in \mathfrak{B}(E_2)$. Writing $\bar{f}(y) := \int f(y, u) \pi(du)$, we see that for any $f \in \mathfrak{D}(A_1)$,

$$\int f(Y(t), u) \pi(du) - \int_0^t \int A_1 f(Y(s), u) \pi(du) ds$$
$$= \bar{f}(Y(t)) - \int_0^t A_2 \bar{f}(Y(s)) ds$$

is an $\{\mathfrak{F}_t^Y\}$-martingale by the definition of Y. Therefore, $(Y, \tilde{\pi})$ is a solution to the filtered martingale problem for $(A_1, \nu_1 \times \pi, \gamma_2)$.

Since A_1 satisfies the conditions of [7] Theorem 2.6, by [7] Theorem 3.2, there exists a solution $(\tilde{X}, \tilde{U})$ to the martingale problem for $(A_1, \nu_1 \times \pi)$ such that $\tilde{X} =^d Y$.

By the definition of Z, for all $f \in \mathfrak{D}(A_1)$ we have

$$f(\gamma_1 \circ Z(t)) - \int_0^t A_1 f(\gamma_1 \circ Z(s)) ds$$
$$= f \circ \gamma_1 (Z(t)) - \int_0^t (A(f \circ \gamma_1)) (Z(s)) ds$$

is an $\{\mathfrak{F}_t^Z\}$-martingale and so an $\{\mathfrak{F}_t^{\gamma_1 \circ Z}\}$-martingale. It follows that $\gamma_1 \circ Z(t)$ is a càdlàg solution to the martingale problem for $(A_1, \nu_1 \times \pi)$.

Since, by assumption, uniqueness holds for this martingale problem, we must have $(\tilde{X}, \tilde{U}) =^d \gamma_1 \circ Z$, and so $Y =^d \gamma_2 \circ \gamma_1 \circ Z$. □

Proof of Proposition 2.2.1. The conditions of Remark 1.3.1 imply that there is a unique solution $\boldsymbol{\Phi} \in \mathbb{Z}^\infty$ to the system of equations

$$\Phi_j(t) = j + \sum_{i \neq j} \int_0^t (\Phi_i(s-) - \Phi_j(s-)) \hat{V}^{(n)}_{\Phi_i(s-), \Phi_j(s-)}(ds) \qquad (2.5)$$

where the $\hat{V}_{ij}^{(n)}$ are given by

$$\hat{V}_{ij}^{(n)}(t) := \sum_{l=1}^{V_{ij}^{(n),\lambda}(t)} \xi_{ijl}$$

for ξ_{ijl} iid random variables, independent of other aspects of the model, that take values 0 and 1 each with probability 1/2.

Moreover, for each t, we have $j \mapsto \Phi_j(t)$ a permutation. Thus, $\mathbf{\Phi}$ starts out as the identity permutation, and when an (ordered) neutral reproductive event occurs at time t between particles indexed by $\Phi_i(t-)$ and $\Phi_j(t-)$, half the time these indices are swapped (so that $\Phi_i(t) = \Phi_j(t-)$ and $\Phi_j(t) = \Phi_i(t-)$), and half the time they are left unchanged.

Now, consider application of Theorem 2.2.2 to the maps

$$(\mathbf{X}^{(n)}, \mathbf{Z}^{(n)}, \mathbf{U}, \mathbf{\Phi}) \xmapsto{\gamma_1} (\mathbf{X}_{\mathbf{\Phi}}^{(n)}, \mathbf{Z}_{\mathbf{\Phi}}^{(n)}, \mathbf{U}_{\mathbf{\Phi}}) \xmapsto{\gamma_2} (\mathbf{X}_{\mathbf{\Phi}}^{(n)}, \mathbf{Z}_{\mathbf{\Phi}}^{(n)})$$

(where by $\mathbf{X}_{\mathbf{\Phi}}^{(n)}$ we mean the process $\mathbf{X}_{\mathbf{\Phi}}^{(n)}(t) = (X_{\Phi_1(t)}^{(n)}(t), X_{\Phi_2(t)}^{(n)}(t), \dots)$ and so forth). Take the spaces to have metrics based on $d(x, y) := \sum_{i=1}^{\infty} \frac{1_{\{x_i \neq y_i\}}}{2^i}$ for the $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{\Phi}$ components and $d(u, v) := \sum_{i=1}^{\infty} \frac{|u_i - v_i|}{2^i}$ for the $\mathbf{U}$ components. Take the generators to be

$$\begin{aligned} Af(x, z, u, \phi) &:= \sum_j B_{n,j}^{\theta} f(x, z, u, \phi) + \sum_j B_j^{\mu} f(x, z, u, \phi) \\ &+ \sum_{\substack{u_i < u_j \\ x_i = x_j \\ z_i \neq z_j}} \lambda \left(\frac{f\big(x, \eta_{ij}(z), u, \phi\big) + f\big(x, \eta_{ij}(z), u, \phi_{\tilde{ij}}\big)}{2} - f(x, z, u, \phi) \right) \\ &+ \sum_{\substack{x_i = x_j \\ z_i \neq z_j}} \sigma(z_i, z_j) \left(f\big(x, \eta_{ij}(z), u, \phi\big) - f(x, z, u, \phi) \right) \end{aligned}$$

with $\eta_{ij}(z) := \eta_j(z|z_i)$ and $\phi_{\tilde{ij}}$ equal to the permutation ϕ with the unique components having values i and j switched;

$$A_1 h(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) := \sum_j B^{\theta}_{n,j} h(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) + \sum_j B^{\mu}_j h(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u})$$
$$+ \sum_{\substack{u_i < u_j \\ x_i = x_j \\ z_i \neq z_j}} \lambda \left(\frac{h\big(\boldsymbol{x}, \eta_{ij}(\boldsymbol{z}), \boldsymbol{u}\big) + h\big(\boldsymbol{x}, \eta_{ij}(\boldsymbol{z}), \boldsymbol{u}_{ij}\big)}{2} - h(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \right)$$
$$+ \sum_{\substack{x_i = x_j \\ z_i \neq z_j}} \sigma(z_i, z_j) \left(h\big(\boldsymbol{x}, \eta_{ij}(\boldsymbol{z}), \boldsymbol{u}\big) - h(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \right)$$

where $\boldsymbol{u}_{ij}$ is equal to $\boldsymbol{u}$ with u_i and u_j swapped; and $A_2 := \tilde{A}_n$, as defined in (1.1). Finally, take π to be iid uniform on $[0, K]$ over all components.

Let $\nu_1 = P\left(\mathbf{X}^{(n)}(0), \mathbf{Z}^{(n)}(0)\right)^{-1}$. Then, $(\mathbf{X}^{(n)}, \tilde{\mathbf{Z}}^{(n)})$ is a solution to the martingale problem for (A_2, ν_1). Moreover, $(\mathbf{X}^{(n)}, \mathbf{Z}^{(n)}, \mathbf{U}, \boldsymbol{\Phi})$ is a solution to the martingale problem for (A, μ) where

$$\mu := \nu_1 \times \pi \times \delta_{id}$$

for δ_{id} the probability mass of 1 on the identity permutation $id\colon \mathbb{Z} \to \mathbb{Z}$. Note $\nu_1 \times \pi = \mu \circ \gamma_1^{-1}$.

To apply Theorem 2.2.2, it remains to show that A_1 satisfies the conditions of [7] Theorem 2.6 and that uniqueness holds for the martingale problem for $(A_1, \nu_1 \times \pi)$. With regards to the conditions on A_1, by Hypothesis 1.3.1, we have a solution Y to the martingale problem (B^{μ}, δ_y) for all y implying that B^{μ} is a pre-generator. We also have B^{μ} closed under multiplication. It follows that A_1 is a pre-generator closed under multiplication. We need only establish that A_1 satisfies a separability hypothesis [7, Hypothesis 2.4] (which holds, for example, if we assume that E is locally compact and that B^{μ} maps continuous functions with compact support to continuous functions with compact support) and that $\mathfrak{D}(A_1)$ separates points.

With regards to the uniqueness condition, we will not show that it holds in this paper, but note that for any solution to $(A_1, \nu_1 \times \pi)$, we can recover—by observing changes to particle levels—a corresponding solution to $(A_n, \nu_1 \times \pi)$ for the generator A_n of (2.4). Thus, uniqueness for this latter problem implies uniqueness for the former.

Application of Theorem 2.2.2 gives

$$(\mathbf{X}^{(n)}, \tilde{\mathbf{Z}}^{(n)}) =^d (\mathbf{X}^{(n)}_{\mathbf{\Phi}}, \mathbf{Z}^{(n)}_{\mathbf{\Phi}})$$

However, as $\mathbf{\Phi}$ is permutation-valued, we have

$$\begin{aligned} \Gamma_\cdot = \sum_j \delta_{(X_j^{(n)}(\cdot), Z_j^{(n)}(\cdot))} &= \sum_j \delta_{(X^{(n)}_{\Phi_j(\cdot)}(\cdot), Z^{(n)}_{\Phi_j(\cdot)}(\cdot))} \\ &=^d \sum_j \delta_{(X_j^{(n)}(\cdot), \tilde{Z}_j^{(n)}(\cdot))} = \tilde{\Gamma}_\cdot \end{aligned}$$

as required. □

2.3 Infinite-Density Neutral Model

Now, let us consider an infinite-density version of the ordered model Υ_n described in Section 2.1 but without selection (taking $\sigma \equiv 0$).

Fix some $n \in \mathbb{N}$, and as in Sections 1.3 and 2.1, let $\nu_0^{(n)}$ be a measure on location/type space $n^{-1}\mathbb{Z} \times E$ with marginal location measure $\nu_0^{(n)}(\cdot \times E) = \sum_{x \in n^{-1}\mathbb{Z}} \frac{1}{n}\delta_x$. Let the initial location, types, and levels $(\mathbf{X}^{(n)}(0), \mathbf{Z}^{(n)}(0), \mathbf{U})$ be some indexing of the points of a Poisson point process with mean measure $\nu_0^{(n)} \times \ell_{\mathbb{R}^+}$. As in Section 2.1, let $\{W_j^{\mathbb{Z}}\}$ be iid simple, symmetric, rate 1 random walks, let $\{N_{ij}^{\lambda}\}$ be iid rate 1 Poisson counting processes, and let $\{Y_{jk}\}$ be iid copies of the mutation process Y such that $\big(\mathbf{X}^{(n)}(0), \mathbf{Z}^{(n)}(0), \mathbf{U}\big)$, $\{W_j^{\mathbb{Z}}\}$, $\{N_{ij}^{\lambda}\}$, and $\{Y_{jk}\}$ are mutually independent.

Consider the location/type/level process $(\mathbf{X}^{(n)}, \mathbf{Z}^{(n)}, \mathbf{U})$ where the location process $\mathbf{X}^{(n)}$ is defined as in equation (1.3) and the level "process" $\mathbf{U}$ is constant. Define neutral event counting processes $\mathbf{V}^{(n),\lambda}$ as in (2.1), and let the type process $\mathbf{Z}^{(n)}$ be given by

$$(\mathbf{Z}^{(n)}, \mathbf{V}, \boldsymbol{\tau}, \boldsymbol{\gamma}, \boldsymbol{\iota}) := \mathfrak{T}(\mathbf{Z}^{(n)}(0), \mathbf{Y}, \mathbf{V}^{(n),\lambda}, 0, 0, 0)$$

in the notation of Remark 1.3.2.

Remark 2.3.1. Note that existence and uniqueness of such a $\mathbf{Z}^{(n)}$ is a more serious question here than in the earlier finite-density models. Fortunately, even though each particle j occupies a site together with an infinite number of other particles, only a small number (specifically, a Poisson number with mean U_j/n) will have levels lower than j's at any one time. Thus, the argument outlined in Remark 1.3.1 applies. It is possible to establish that the V_j have no explosions and that the collection of all (lower level) particles that might have influenced the value of $Z_j^{(n)}(T)$ for any fixed T is finite. The existence and uniqueness then follow.

In contrast, note that if we had included selection in the form of the potential selective event counting processes given by (2.2), this argument would not apply. Since selective events are unordered, every particle would be subject to infinitely many potential selective type changes in every finite time interval. In general, there would be no solution $\mathbf{Z}^{(n)}$.

We will consider how selection may be introduced into infinite-density models in Section 3.2 of the next chapter.

Note that, in the case without mutation, the type process $\mathbf{Z}^{(n)}$ satisfies

$$Z_j^{(n)}(t) = Z_j^{(n)}(0) + \sum_i 1_{\{U_i<U_j\}} \int_0^t \left(Z_i^{(n)}(s-) - Z_j^{(n)}(s-)\right) dV_{ij}^{(n),\lambda}(s)$$

(though the indicator $1_{\{U_i<U_j\}}$ is redundant as it already included in the definition of the $V_{ij}^{(n),\lambda}$).

The generator $A_{n,\lambda}$ for $(\mathbf{X}^{(n)}, \mathbf{Z}^{(n)}, \mathbf{U})$ is the A_n of (2.4) but with $\sigma \equiv 0$, giving

$$\begin{aligned} A_{n,\lambda} f(\mathbf{x}, \mathbf{z}, \mathbf{u}) := \sum_j B_{n,j}^{\theta} f(\mathbf{x}, \mathbf{z}, \mathbf{u}) + \sum_j B_j^{\mu} f(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ + \sum_{\substack{i \neq j \\ x_i = x_j}} \lambda 1_{\{u_i<u_j\}} \left(f\big(\mathbf{x}, \eta_j(\mathbf{z}|z_i), \mathbf{u}\big) - f(\mathbf{x}, \mathbf{z}, \mathbf{u})\right) \end{aligned}$$

For initial location, type, and level vectors $(\mathbf{X}_0, \mathbf{Z}_0, \mathbf{U})$, we will write

$$\Upsilon_{n,\lambda}(\mathbf{X}_0, \mathbf{Z}_0, \mathbf{U}) := (\mathbf{X}^{(n)}, \mathbf{Z}^{(n)}, \mathbf{U})$$

The following proposition is immediate from the construction of the models Υ_n and $\Upsilon_{n,\lambda}$.

Proposition 2.3.1. *In the neutral case ($\sigma \equiv 0$), let*

$$(\mathbf{X}, \mathbf{Z}, \mathbf{U}) = \Upsilon_{n,\lambda}\big(\mathit{Poisson}(\nu_0^{(n)} \times \ell_{\mathbb{R}^+})\big)$$

be the infinite density model of the present section.

For any $K > 0$, if we take

$$(\mathbf{X}', \mathbf{Z}', \mathbf{U}') = \Upsilon_n\big(\mathit{Poisson}(K\nu_0^{(n)}), K\big)$$

a realization of the K-density model of Section 2.1, then we have

$$\sum_j \delta_{(X_j', Z_j', U_j')} =^d \sum_{U_j \le K} \delta_{(X_j, Z_j, U_j)}$$

Combining the results of Propositions 2.2.1 and 2.3.1, we have that—in the absence of selection—the infinite-density, ordered model $\Upsilon_{n,\lambda}$ of the present section, cut off at level K, has the same distribution, up to particle indexing, as the K-density, ordered model Υ_n of Section 2.1 and model Υ_n has the same empirical location/type distribution as the K-density, symmetric model $\tilde{\Upsilon}_n$ of Section 1.3. That is, the infinite-density, ordered model of the present section simultaneously embeds (symmetric) neutral Moran stepping stone models of all densities $K > 0$.

Now, we will turn our attention away from lattice models. In the next chapter, we will consider an ordered, infinite-density variation of the local-time interaction model $\tilde{\Upsilon}_0$ of Section 1.4. It should be noted, however, that for many of the results we obtain in Chapter 3, analogous results hold for the model of the present section. The construction, in Section 3.2, of an infinite-density Brownian model *with selection* could be carried out for stepping-stone models, generalizing $\Upsilon_{n,\lambda}$ to the selective case. Similarly, the coupling technique of Section 3.3 can be applied to lattice models and might be used as an alternative to the generator-based method of Section 2.2. Of somewhat more interest, the limiting location/type measure obtained in Section 3.4 has an analogue in the present case and an analogous Poisson structure result to that of Theorem 3.5.3 will hold.

Chapter 3

Infinite-Density Brownian Model

> "Rhythm is the basis of life, not steady forward progress. The forces of creation, destruction, and preservation have a whirling, dynamic interaction."
>
> — *Kabbalah (circa 100–1000 A.D.)*

3.1 Ordered Interacting Brownian Motions

In this chapter, we study an infinite-density, ordered Brownian motion model with particle interactions determined by local times.

In Section 1.4, we examined the limit of the (symmetric) Moran stepping stone model $\tilde{\Upsilon}_n$ as the site density n increased with the particle density K held constant. Recall that the particle location random walks converged to independent Brownian motions and, with the interaction parameter scalings $\lambda = \lambda_0 n$ and $\sigma = \sigma_0 n$, the interactions between pairs of particles converged to Poisson counting processes with clocks proportional to the local times at zero of the distances between the particles. We called the resulting model, with an appropriately defined type process, model $\tilde{\Upsilon}_0$.

In this section, we begin with an ordered version Υ_0 of the K-density, Brownian model $\tilde{\Upsilon}_0$. In Section 3.2, we consider how to extend this model to an infinite density model Υ_∞ *with selection*. In Section 3.3, we show by means of an "intermediate" model construction that the K-density, ordered model Υ_0 may be coupled to the K-density, symmetric model of Section 1.4.

In Sections 3.4 and 3.5, we establish that there is a time-indexed collection of measure-valued random variables ν_t associated with Υ_∞. Conditioned on ν_t, the points of Υ_∞ at time t are Poisson with measure $\nu_t \times \ell_{\mathbb{R}^+}$ in location/type/level space. In essence, the measure ν_t is the measure-valued limiting diffusion process for the locations and types of the ordered model, and—by the coupling results of Section 3.3—it is the measure-valued diffusion limit of the symmetric model $\tilde{\Upsilon}_0$ as $K \to \infty$.

We study the process ν_t more carefully in the final sections, proving a martingale characterization in Section 3.6 and establishing a tightness result—and so showing it almost surely has vaguely continuous paths—in Section 3.7. Finally, Section 3.8 contains the proof of Theorem 3.6.2, an involved calculation of the quadratic variations of the martingales studied in Section 3.6.

Let us begin by constructing an ordered version Υ_0 of the model $\tilde{\Upsilon}_0$ of Section 1.4. Fix a particle density $K > 0$. Let ν_0 be some measure on location/type space $\mathbb{R} \times E$ with marginal measure $\nu_0(\cdot \times E)$ equal to $\ell_{\mathbb{R}}$, and let the initial locations, types, and levels $(\mathbf{X}(0), \mathbf{Z}(0), \mathbf{U})$ be some indexing of the points of a Poisson point process with mean measure $\nu_0 \times \ell_{[0,K]}$. Let Y be the mutation process with generator B^μ described in (1.2). Let $\{W_j : j \in \mathbb{N}\}$ be iid standard Brownian motions, let $\{N^\lambda_{ij}, \bar{N}^\sigma_{ij} : i \neq j \in \mathbb{N}\}$ be iid, rate 1 Poisson counting processes, let $\{Y_{jk} : j \in \mathbb{N}, k \in \mathbb{Z}^+\}$ be iid copies of Y, and let $\{\zeta_{jk} : j \in \mathbb{N}, k \in \mathbb{Z}^+\}$ be iid uniform on $[0,1]$ such that $\big(\mathbf{X}(0), \mathbf{Z}(0), \mathbf{U}\big)$, $\{W_j\}$, $\{N^\lambda_{ij}\}$, $\{\bar{N}^\sigma_{ij}\}$, $\{Y_{jk}\}$, and $\{\zeta_{jk}\}$ are mutually independent.

Define the location process $\mathbf{X}$ by

$$X_j := X_j(0) + \sqrt{\theta} W_j$$

For each ordered particle pair (i,j) with $i \neq j$, let $L_{ij} := L^0_t(X_i - X_j)$ be the local time at zero of $X_i - X_j$, and define the neutral and potential selective event counting process by

$$V_{ij}^{\lambda} := 1_{\{u_i < u_j\}} N_{ij}^{\lambda}\left(\frac{\lambda}{2\theta} L_{ij}(\cdot)\right)$$
$$\bar{V}_{ij}^{\sigma} := \bar{N}_{ij}^{\sigma}\left(\frac{\bar{\sigma}}{2\theta} L_{ij}(\cdot)\right)$$

In the notation of Remark 1.3.2, define the type process $\mathbf{Z}$ by

$$(\mathbf{Z}, \mathbf{V}, \boldsymbol{\tau}, \boldsymbol{\gamma}, \boldsymbol{\iota}) := \mathfrak{T}(\mathbf{Z}(0), \mathbf{Y}, \mathbf{V}^{\lambda}, \bar{\mathbf{V}}^{\sigma}, \zeta, \bar{\sigma}^{-1}\sigma)$$

We will write the above model as Υ_0 so that

$$\Upsilon_0(\mathbf{X}_0, \mathbf{Z}_0, \mathbf{U}) := (\mathbf{X}, \mathbf{Z}, \mathbf{U})$$

and we will define the empirical location/type process ξ^K by

$$\xi_t^K := \sum_j \delta_{(X_j(t), Z_j(t))} \tag{3.1}$$

This model has a simple geometric interpretation. We begin with a Poisson collection of particles uniformly distributed in location/level space $\mathbb{R} \times [0, K]$. (Projected onto the location component, of course, the particles form a K-density collection occupying the real line.) The particles engage in independent, horizontal Brownian motions, moving through location space while their levels remain constant. Neutral events between each unordered pair of particles occur according to a Poisson counting process driven by the local time of the horizontal distance between the pair. When such an event occurs, the particles are necessarily at the same location; the particle with the higher level "looks down" and copies the type of the lower-level particle. Selection events occur according to a Poisson counting process driven by an integral against the local time whose integrand takes into account the types of the particles. Unlike neutral events, selection events are unordered, so it is possible for either a lower-level particle to "look up" or a higher-level particle to "look down" to copy a type. Between interactions, the particle types are independently subject to mutation.

3.2 Infinite-Density Model with Selection

Fix $K > 0$, and consider the model of the last section

$$(\mathbf{X}, \mathbf{Z}, \mathbf{U}) = \Upsilon_0(\mathbf{X}_0, \mathbf{Z}_0, \mathbf{U})$$

for $(\mathbf{X}_0, \mathbf{Z}_0, \mathbf{U}) \sim \mathit{Poisson}(\nu_0 \times \ell_{[0,K]})$. For each $j \in \mathbb{N}$, define the counting process

$$\bar{V}_j^\sigma := \sum_{i \neq j} \bar{V}_{ij}^\sigma$$

which counts the total number of potential selective events directly affecting particle j. Observe that

$$\bar{V}_j^\sigma = \bar{N}_j^\sigma \left(\frac{\bar{\sigma}}{2\theta} \sum_{i \neq j} L_{ij}(\cdot) \right)$$

for $\{\bar{N}_j^\sigma\}$ iid rate one Poisson counting processes independent of other aspects of the model. Our present goal is to characterize this time change for an "arbitrary" particle j as the particle density K increases to infinity.

For these purposes, let us condition on $\mathbf{X}_0$ having a particle at some arbitrary location $a \in \mathbb{R}$. For convenience, let us take this particle to have index one, so that $X_{0,1} = a$. By the properties of Poisson processes, we have

$$\sum_{i \neq 1} \delta_{X_{0,i}} \sim \mathit{Poisson}(\ell_{\mathbb{R}})$$

Fix $t > 0$. Define

$$R^K := \frac{\bar{\sigma}}{2\theta} \sum_{i \neq 1} L_{i,1}(t)$$

This is the time change associated with $\bar{V}_1^\sigma$. For each $\alpha > 0$, let us define

$$R_\alpha^K := \frac{\bar{\sigma}}{2\theta} \sum_{i \neq 1} 1_{\{|X_{0,i} - X_{0,1}| < \alpha\}} L_{i,1}(t)$$

the total contribution to R^K made by particles whose initial locations are within α of particle one's initial location a. Observe that

$$\sum_{i \neq 1} 1_{\{|X_{0,i} - X_{0,1}| < \alpha\}} \delta_{(X_{0,i} - X_{0,1}, W_i, W_1)} =^d \sum_{i=1}^{N} \delta_{(\hat{u}_i, \hat{W}_i, \hat{W}_0)}$$

where N is Poisson with mean $2K\alpha$, $\hat{U}_i$ is uniform $[-\alpha, \alpha]$, and $\hat{W}_0$ and $\hat{W}_i$ are standard Brownian motions with N, $\hat{U}_i$, $\hat{W}_0$, and $\hat{W}_i$ mutually independent. It follows that

$$R_\alpha^K =^d \frac{\bar{\sigma}}{2\theta} \sum_{i=1}^{N} L_t^0(\hat{U}_i + \sqrt{\theta}(\hat{W}_i - \hat{W}_0))$$

The following property of local times will be useful here

Lemma 3.2.1. *Let X_t be a semimartingale with continuous paths and local time $L_t^a(X)$ at point a. For any $\sigma > 0$ and any b, the process $\sigma X_t + b$ has local time $L_t^a(\sigma X + b)$ at point a given by*

$$L_t^a(\sigma X + b) = \sigma L_t^{\frac{a-b}{\sigma}}(X)$$

Proof. By Corollary 3 of Theorem IV.51 of [10], we have

$$\begin{aligned}
L_t^a(\sigma X + b) &= |\sigma X_t - a + b| - |\sigma X_0 - a + b| - \sigma \int_0^t \operatorname{sgn}(\sigma X_s - a + b) dX_s \\
&= \sigma \left\{ \left|X_t - \tfrac{a-b}{\sigma}\right| - \left|X_0 - \tfrac{a-b}{\sigma}\right| - \int_0^t \operatorname{sgn}\left(X_s - \tfrac{a-b}{\sigma}\right) dX_s \right\} \\
&= \sigma L_t^{\frac{a-b}{\sigma}}(X)
\end{aligned}$$

giving the result. ☐

By Lemma 3.2.1, we see that

$$\mathrm{E}\, R_\alpha^K = \bar{\sigma} K \int_{-\alpha/\sqrt{2\theta}}^{\alpha/\sqrt{2\theta}} \mathrm{E}\, l_t^a da$$

where l_t^a is the local time at a of a standard Brownian motion.

The following lemma and its corollary provide the missing detail.

Lemma 3.2.2. *Let W_t be a standard Brownian motion with $W_0 = a > 0$. Then, the local time l_t at zero satisfies*

$$\mathrm{E}\, l_t = \sqrt{\frac{2t}{\pi}} e^{-\frac{a^2}{2t}} - 2a\Phi\left(-\frac{a}{\sqrt{t}}\right)$$

Proof. Tanaka's Formula [10] states that

$$l_t = |W_t| - |W_0| - \beta_t$$

where β_t is a Brownian motion with $\beta_0 = 0$. Clearly then

$$\begin{aligned} \mathrm{E}\, l_t &= \mathrm{E}\,|W_t| - a \\ &= \sqrt{\frac{2t}{\pi}} e^{-\frac{a^2}{2t}} - 2a\Phi\left(-\frac{a}{\sqrt{t}}\right) \end{aligned}$$

□

Corollary 3.2.3. *For l_t^a the local time at a of a standard Brownian motion, we have*

$$\int_{-\infty}^{\infty} \mathrm{E}\, l_t^a da = t$$

Proof. Note that

$$\begin{aligned} \int_{-M}^{M} \mathrm{E}\, l_t^a\, da &= 2\int_0^M \left(\sqrt{\frac{2t}{\pi}} e^{-\frac{a^2}{2t}} - 2a\Phi\left(-\frac{a}{\sqrt{t}}\right)\right) da \\ &= t\Phi\left(-\frac{M}{\sqrt{t}}, \frac{M}{\sqrt{t}}\right) - M^2\Phi\left(-\frac{M}{\sqrt{t}}, \frac{M}{\sqrt{t}}\right)^c + Me^{-M^2/2t}\sqrt{\frac{2t}{\pi}} \\ &\to t \end{aligned}$$

Actually, this result also follows directly from Corollary 2 of Theorem IV.51 of [10], since

$$\mathrm{E}\int_{-\infty}^{\infty} l_t^a\, da = \mathrm{E}[W]_t = t$$

□

Since $R_\alpha^K \nearrow R^K$ as $\alpha \to \infty$, it follows by the corollary that

$$\mathrm{E}\, R^K = \bar{\sigma} K t \tag{3.2}$$

Similarly, note that

$$
\begin{aligned}
\mathrm{E}(R_\alpha^K)^2 &= \frac{\bar{\sigma}^2}{4\theta^2}\,\mathrm{E}\left[\sum_{i=1}^{N} L_t^0(\hat{U}_i + \sqrt{\theta}(\hat{W}_i - \hat{W}_0))\right]^2 \\
&= \frac{\bar{\sigma}^2}{4\theta^2}\Big(2K\alpha\,\mathrm{E}\left[L_t^0\left(\hat{U}_1 + \sqrt{\theta}(\hat{W}_1 - \hat{W}_0)\right)\right]^2 \\
&\qquad + (2K\alpha)^2\,\mathrm{E}\Big[L_t^0\left(\hat{U}_1 + \sqrt{\theta}(\hat{W}_1 - \hat{W}_0)\right) \\
&\qquad\qquad \cdot L_t^0\left(\hat{U}_2 + \sqrt{\theta}(\hat{W}_2 - \hat{W}_0)\right)\Big]\Big) \\
&= \frac{\bar{\sigma}^2 K\alpha}{\theta}\left(\mathrm{E}(l_{1,t})^2 + 2K\alpha\,\mathrm{E}\,l_{1,t}l_{2,t}\right)
\end{aligned}
\tag{3.3}
$$

where

$$
\begin{aligned}
l_{1,t} &:= L_t^0\left(\frac{1}{\sqrt{2\theta}}\hat{U}_1 + \frac{1}{\sqrt{2}}(\hat{W}_0 - \hat{W}_1)\right) \\
l_{2,t} &:= L_t^0\left(\frac{1}{\sqrt{2\theta}}\hat{U}_2 + \frac{1}{\sqrt{2}}(\hat{W}_0 - \hat{W}_2)\right)
\end{aligned}
\tag{3.4}
$$

are local times of dependent standard Brownian motions with starting points uniformly random on $\left[-\alpha/\sqrt{2\theta}, \alpha/\sqrt{2\theta}\right]$.

To bound the variance and covariance of these local times, we will need several results. We begin with a theorem that bounds the second moment of the local time of a Brownian motion whose starting point is uniformly distributed on a large set.

Lemma 3.2.4. *Let* $X := X_0 + W$ *for* X_0 *uniform on* $[-M, M]$ *independent of the standard Brownian motion* W*. Let* L *be the local time at zero of* X*. Then*

$$\mathrm{E}(L_t)^2 \le \frac{\nu_t}{M}\sqrt{\frac{2t}{\pi}}$$

where $\nu_t := \mathrm{E}(l_t)^2 < \infty$ *is the second moment of the local time at 0 of a standard Brownian motion.*

Proof. Let $a > 0$. Define $\tau_a := \inf\{t : W_t = a\}$ to be the hitting time of a. Writing l^a as the local time at a of W, by the strong Markov property of W, we have

$$\mathrm{E}(l_t^a)^2 = \mathrm{E}\left[\mathrm{E}\left[(l_t^a)^2 \mid \mathfrak{F}_{\tau_a}^W\right]\right] \le \nu_t\,\mathrm{E}\,1_{\{\tau_a < t\}} = \nu_t\,\mathrm{P}\left(\left|\sqrt{t}Z\right| > a\right)$$

for Z a standard normal random variable.

It follows that

$$\begin{aligned} E(L_t)^2 &= \frac{1}{2M}\int_{-M}^{M} E(l_t^a)^2 da \\ &\leq \frac{\nu_t}{M}\sqrt{t}\int_0^{M/\sqrt{t}} P(|Z| > a)da \\ &\leq \frac{\nu_t}{M}\sqrt{t}\, E\,|Z| \end{aligned}$$

giving the result. ☐

We will now need the following technical lemma.

Lemma 3.2.5. *Let* $U = \int X dY$ *for* $|X| \leq C$ *and* Y *a continuous, square integrable martingale. For any* $p \geq 2$*, we have*

$$E \sup_{s \leq t} |U_s|^p \leq b_p C^p\, E[Y,Y]_t^{p/2}$$

where $b_p = \left\{ q^p \left(\frac{p(p-1)}{2} \right) \right\}^{p/2}$.

Proof. As X is bounded, U is a continuous, square integrable martingale. Therefore, by Burkhölder's Inequality [10, Theorem IV.54], we have

$$\begin{aligned} E \sup_{s \leq t} |U_s|^p &\leq b_p\, E \left(\int_0^t X_s^2 d[Y,Y]_s \right)^{p/2} \\ &\leq b_p C^p\, E[Y,Y]_t^{p/2} \end{aligned}$$

☐

Next, we show uniform integrability for a certain sequence of integrals.

Lemma 3.2.6. *For* $\delta > 0$*, let* X *and* Y *be continuous, square-integrable martingales such that* $E[X,X]_t^{1+\delta}$*,* $E[Y,Y]_t^{1+\delta}$*,* $E\, X_0^{2+2\delta}$*, and* $E\, Y_0^{2+2\delta}$ *are finite. Define*

$$Z_n := n^2 \int_0^t 1_{\{X_s \in (0,1/n)\}} d[X]_s \int_0^t 1_{\{Y_s \in (0,1/n)\}} d[Y]_s$$

Then, we have $\sup_n E\, Z_n^{1+\delta} < \infty$*. In particular, the* Z_n *are uniformly integrable.*

Proof. Fix n. Let ϕ_a be defined by

$$\phi_a(x) := \begin{cases} -(1-a)x, & x \le 0; \\ \frac{n^2x^3}{3a} - (1-a)x & 0 \le x \le \frac{a}{n}; \\ n\left(x - \frac{1}{n}\right)x + \frac{a^2}{3n}, & \frac{a}{n} \le x \le \frac{1-a}{n}; \\ \frac{n^2\left(\frac{1}{n}-x\right)^3}{3a} + (1-a)\left(x - \frac{1}{n}\right), & \frac{1-a}{n} \le x \le \frac{1}{n}; \\ (1-a)\left(x-\frac{1}{n}\right), & \frac{1}{n} \le x. \end{cases}$$

Note the following properties of ϕ_a:

- $\phi_a \in C^2(\mathbb{R})$;
- $|\phi_a(x)| \le |x|$;
- $|\phi_a'(x)| \le 1$;
- $\phi_a''(x) \ge 0$ increases to a limit $2n1_{(0,\frac{1}{n})}(x)$ as $a \to 0$.

Let $p = 2 + 2\delta$. By Lemma 3.2.5,

$$\mathrm{E}\left|\int_0^t \phi_a'(X_s)\,dX_s\right|^p \le b_p\,\mathrm{E}[X,X]_t^{1+\delta}$$

Thus, by the Monotone Convergence Theorem, Itô's formula, and the Minkowski Inequality, we have

$$\begin{aligned} \left\|\int_0^t n1_{\{X_s\in(0,1/n)\}}d[X]_s\right\|_p &= \lim_{a\to 0}\left\|\frac{1}{2}\int_0^t \phi_a''(X_s)\,d[X]_s\right\|_p \\ &= \lim_{a\to 0}\left\|\phi_a(X_t) - \phi_a(X_0) - \int_0^t \phi_a'(X_s)\,dX_s\right\|_p \\ &\le \|X_t\|_p + \|X_0\|_p + \left(b_p\,\mathrm{E}[X,X]_t^{1+\delta}\right)^{1/p} \end{aligned}$$

An analogous result holds for Y, so we may conclude by the Hölder Inequality that

$$\begin{aligned} \mathrm{E}\,Z_n^{1+\delta} \le \left(\|X_t\|_p + \|X_0\|_p + \left(b_p\,\mathrm{E}[X,X]_t^{1+\delta}\right)^{1/p}\right)^{1+\delta} \\ \left(\|Y_t\|_p + \|Y_0\|_p + \left(b_p\,\mathrm{E}[Y,Y]_t^{1+\delta}\right)^{1/p}\right)^{1+\delta} \end{aligned}$$

The expression on the right is independent of n. By hypothesis, $\|X_0\|_p$, $\|Y_0\|_p$, $E[X,X]_t^{1+\delta}$, and $E[Y,Y]_t^{1+\delta}$ are finite, and by Burkhölder's Inequality, this implies $\|X_t\|_p$ and $\|Y_t\|_p$ are finite. Therefore, we have the result. $\square$

Finally, we are ready to give the following result, bounding the covariance of local times of correlated Brownian motions having uniform starting points on a large set.

Lemma 3.2.7. *Let (X, Y) be a two-dimensional Brownian motion with X_0 and Y_0 iid uniform on $[-M, M]$ and quadratic variations $[X, X]_t = [Y, Y]_t = t$ and $[X, Y] = \rho t$ for some $\rho \in [-1, 1]$. Let L and K be the local times at zero of X and Y respectively. Then,*

$$1 - \frac{t}{M^2} \leq \frac{4M^2}{t^2} \, E \, L_t K_t \leq 1$$

Proof. Fix t. By Corollary 3 of Theorem IV.55 of [10], we have

$$L_t K_t = \lim_{n \to \infty} Z_n$$

where

$$Z_n = n^2 \int_0^t \int_0^t 1_{(0, \frac{1}{n})}(X_s) 1_{(0, \frac{1}{n})}(Y_r) \, ds \, dr$$

By Lemma 3.2.6, the Z_n are uniformly integrable, so by Tonelli's theorem, we have

$$\begin{aligned} E \, L_t K_t &= \lim_{n \to \infty} E \, Z_n \\ &= \lim_{n \to \infty} \int_0^t \int_0^t n^2 \, P\left(X_s \in (0, 1/n), Y_r \in (0, 1/n)\right) ds \, dr \\ &= \lim_{n \to \infty} \int_0^t \int_0^t ds \, dr \iint_{[-M,M]^2} \frac{dx \times dy}{4M^2} n^2 \, P\left(0 \leq Z_1 \leq \frac{1}{n}, 0 \leq Z_2 \leq \frac{1}{n}\right) \end{aligned}$$

for $(Z_1, Z_2)' \sim N\left((x, y)', \Sigma\right)$ where $\Sigma = \left(\begin{smallmatrix} s & \rho(s \wedge r) \\ \rho(s \wedge r) & r \end{smallmatrix}\right)$.

Note that

$$n^2 \, P\left(Z_1 \in (0, 1/n), Z_2 \in (0, 1/n)\right) \nearrow \phi\left((x, y)', \Sigma\right)$$

for $\phi(\cdot,\Sigma)$ the density of $N(0,\Sigma)$, so by the Monotone Convergence Theorem, we have

$$\mathrm{E}\, L_t K_t = \int_0^t \int_0^t ds\, dr \iint_{[-M,M]^2} \frac{dx \times dy}{4M^2} \phi\left((x,y)',\Sigma\right)$$

However, by Markov's Theorem, for $(U,V)' \sim N(0,\Sigma)$, we have

$$\begin{aligned}\iint_{\mathbb{R}^2\setminus[-M,M]^2} dx \times dy\ \phi\left((x,y)',\Sigma\right) &\le P(|U+V| > 2M) \\ &\le \frac{\mathrm{E}(U+V)^2}{4M^2} \\ &\le \frac{t}{M^2}\end{aligned}$$

for all $s, r \le t$. Therefore, we have

$$1 - \frac{t}{M^2} \le \iint_{[-M,M]^2} dx \times dy \ \ \phi\left((x,y)',\Sigma\right) \le 1$$

and the result follows. □

Now, applying Lemmas 3.2.4 and 3.2.7 to the task at hand, we see that for the $l_{1,t}$ and $l_{2,t}$ of (3.4), we have

$$0 \le \mathrm{E}(l_{1,t})^2 \le \frac{2\nu_t}{\alpha}\sqrt{\frac{\theta t}{\pi}}$$

and

$$\frac{\theta t^2}{2\alpha^2}\left(1 - \frac{2\theta t}{\alpha^2}\right) \le \mathrm{E}\, l_{1,t} l_{2,t} \le \frac{\theta t^2}{2\alpha^2}$$

Therefore, by (3.3), we have

$$\bar{\sigma}^2 t^2 K^2 \left(1 - \frac{2\theta t}{\alpha^2}\right) \le \mathrm{E}(R^K_\alpha)^2 \le \bar{\sigma}^2 t^2 K^2 + 2\bar{\sigma}^2 K \nu_t \sqrt{\frac{t}{\pi\theta}}$$

and so, letting $\alpha \to \infty$, we have

$$\bar{\sigma}^2 t^2 K^2 \le \mathrm{E}(R^K)^2 \le \bar{\sigma}^2 t^2 K^2 + 2\bar{\sigma}^2 K \nu_t \sqrt{\frac{t}{\pi\theta}}$$

If we take some per-particle selection rate function and upper bound $0 \le \sigma_0(\cdot,\cdot) \le \bar{\sigma}_0$ and let $\sigma := \sigma_0/K$ and $\bar{\sigma} := \bar{\sigma}_0/K$, then

$$\bar{\sigma}_0^2 t^2 \le \mathrm{E}(R^K)^2 \le \bar{\sigma}_0^2 t^2 + 2\bar{\sigma}_0^2 \frac{\nu_t}{K}\sqrt{\frac{t}{\pi\theta}} \xrightarrow[K\to\infty]{} \bar{\sigma}_0^2 t^2$$

Together with (3.2), this implies that

$$R^K \xrightarrow[K\to\infty]{L^2} \bar{\sigma}_0 t$$

That is, the per-particle potential selective event processes $\bar{V}_j^\sigma$ are, in the infinite-density limit, iid Poisson counting processes with constant rate $\bar{\sigma}_0$.

This motivates the following construction of an infinite-density, ordered Brownian motion model with per-particle, constant-rate Poisson potential selective event counting processes.

As in the previous section, let ν_0 be some measure on location/type space $\mathbb{R} \times E$ with marginal measure $\nu_0(\cdot \times E)$ equal to $\ell_{\mathbb{R}}$, and let the initial locations, types, and levels $(\mathbf{X}(0), \mathbf{Z}(0), \mathbf{U})$ be some indexing of the points of a Poisson point process with mean measure $\nu_0 \times \ell_{\mathbb{R}^+}$. Let $\{W_j : j \in \mathbb{N}\}$ be iid standard Brownian motions, let $\{N_{ij}^\lambda : i \neq j \in \mathbb{N}\}$ and $\{\bar{N}_j^\sigma : j \in \mathbb{N}\}$ be iid, rate 1 Poisson counting processes, let $\{Y_{jk} : j \in \mathbb{N}, k \in \mathbb{Z}^+\}$ be iid copies of the mutation process Y, and let $\{\zeta_{jk}, \eta_{jk} : j \in \mathbb{N}, k \in \mathbb{Z}^+\}$ be iid uniform on $[0,1]$ such that $\big(\mathbf{X}(0), \mathbf{Z}(0), \mathbf{U}\big)$, $\{W_j\}$, $\{N_{ij}^\lambda\}$, $\{\bar{N}_j^\sigma\}$, $\{Y_{jk}\}$, $\{\zeta_{jk}\}$, and $\{\eta_{jk}\}$ are mutually independent.

Define the location process $\mathbf{X}$ by

$$X_j := X_j(0) + \sqrt{\theta} W_j$$

let $L_{ij} = L_t^0(X_i - X_j)$ be the local time at zero of $X_i - X_j$, and define the per-pair neutral event counting process by

$$V_{ij}^\lambda(t) := 1_{\{U_i < U_j\}} N_{ij}^\lambda\left(\frac{\lambda}{2\theta} L_{ij}(t)\right)$$

For each j, define the per-particle potential selective event counting process by

$$\bar{V}_j^\sigma(t) := \bar{N}_j^\sigma(\bar{\sigma} t)$$

and define the counting process

$$V_j := \sum_{i \neq j} V_{ij}^{\lambda} + \bar{V}_j^{\sigma}$$

which counts all neutral and potential selective reproduction events directly affecting particle j. For $k \in \mathbb{N}$, let τ_{jk} be the kth jump time of V_j, and define

$$\gamma_{jk} := \sum_{i \neq j} i \Delta V_{ij}^{\lambda}(\tau_{jk})$$

$$\iota_{jk} := \sum_{i \neq j} \Delta V_{ij}^{\lambda}(\tau_{jk})$$

Almost surely, the summations γ_{jk} and ι_{jk} have at most one non-zero term. If τ_{jk} corresponds to a neutral event ($\iota_{jk} = 1$), then γ_{jk} gives the index of the particle with which particle j interacts. (In the case that τ_{jk} represents a potential selective event, we have $\iota_{jk} = \gamma_{jk} = 0$.)

With the conventions $\tau_{j,0} := 0$, $\gamma_{j,0} := j$, $\iota_{j,0} := 1$, and $Z_j(0-) := Z_j(0)$, we define the type process Z to be the unique solution of the system of equations

$$Z_j(t) = Y_{jk}\left(\phi\left(\iota_{jk} Z_{\gamma_{jk}}(\tau_{jk}-) + (1-\iota_{jk})\psi_{jk}, Z_j(\tau_{jk}-), \iota_{jk}, \zeta_{jk}\right), t - \tau_{jk}\right), \quad \tau_{jk} \leq t < \tau_{j,k+1}$$

where ϕ is defined as in (1.7) and ψ_{jk} are E-valued solutions to the equations

$$\psi_{jk} = \psi(\xi_{\tau_{jk}-}, X_j(\tau_{jk}-), \eta_{jk}) \tag{3.5}$$

where ψ is some (deterministic) measurable function and ξ is the empirical location/type/level process given by

$$\xi_t := \sum_j \delta_{(X_j(t), Z_j(t), u_j)} \tag{3.6}$$

Remark 3.2.1. In the neutral case ($\sigma \equiv 0$), the considerations of Remark 2.3.1 are relevant. In fact, for the initial distribution described here, the techniques used in the proofs of Section 3.3 may be used to establish existence and uniqueness of $\mathbf{Z}$, and the formulation above is equivalent to taking

$$(\mathbf{Z}, \mathbf{V}, \boldsymbol{\tau}, \boldsymbol{\gamma}, \boldsymbol{\iota}) := \mathfrak{T}(\mathbf{Z}(0), \mathbf{Y}, \mathbf{V}^{\lambda}, 0, 0, 0)$$

in the notation of Remark 1.3.2.

Remark 3.2.2. As in Remark 1.3.2, we will introduce a special notation for the type-definition procedure used above. If we write

$$(\boldsymbol{z}, \xi, \boldsymbol{\nu}, \boldsymbol{\tau}, \boldsymbol{\gamma}, \boldsymbol{\iota}, \boldsymbol{\psi}) := \mathfrak{T}'(\boldsymbol{z}_0, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y}, \boldsymbol{\nu}^{\lambda}, \bar{\boldsymbol{\nu}}^{\sigma}, \boldsymbol{\zeta}, \boldsymbol{\eta}, \bar{\sigma}^{-1}\sigma)$$

we mean to indicate that we are defining $\boldsymbol{\nu}$ by

$$\nu_j := \sum_{i \neq j} \nu_{ij}^{\lambda} + \bar{\nu}_j^{\sigma}$$

defining τ_{jk} to be the ordered jump times of ν_j; defining $\boldsymbol{\gamma}$ and $\boldsymbol{\iota}$ by

$$\gamma_{jk} := \sum_{i \neq j} i \Delta \nu_{ij}^{\lambda}(\tau_{jk})$$
$$\iota_{jk} := \sum_{i \neq j} \Delta \nu_{ij}^{\lambda}(\tau_{jk})$$

adopting the conventions $\tau_{j,0} := 0$, $\gamma_{j,0} := j$, $\iota_{j,0} := 1$, and $z_j(0-) := z_j(0)$; and finally defining $\boldsymbol{z}$, ξ, and $\boldsymbol{\psi}$ as the unique solution to

$$z_j(t) = y_{jk}\left(\phi\left(\iota_{jk} z_{\gamma_{jk}}(\tau_{jk}-) + (1 - \iota_{jk})\psi_{jk}, z_j(\tau_{jk}-), \iota_{jk}, \zeta_{jk}\right), t - \tau_{jk}\right), \quad \tau_{jk} \leq t < \tau_{j,k+1}$$

with

$$\psi_{jk} = \psi(\xi_{\tau_{jk}-}, x_j(\tau_{jk}-), \eta_{jk})$$

and

$$\xi_t = \sum_j \delta_{(x_j(t), z_j(t), u_j)}$$

In particular, the above formulation could have been written

$$(\mathbf{Z}, \xi, \mathbf{V}, \boldsymbol{\tau}, \boldsymbol{\gamma}, \boldsymbol{\iota}, \boldsymbol{\psi}) := \mathfrak{T}'(\mathbf{Z}(0), \mathbf{X}, \mathbf{U}, \mathbf{Y}, \mathbf{V}^{\lambda}, \bar{\mathbf{V}}^{\sigma}, \boldsymbol{\zeta}, \boldsymbol{\eta}, \bar{\sigma}^{-1}\sigma) \tag{3.7}$$

We have not specified the form of the function ψ, but the implication is that ψ uses the random input η_{jk} to pick a type from the type distribution of particles "at" particle j's present location $X_j(\tau_{jk})$. Of course, almost surely, there are no particles "at" $X_j(\tau_{jk})$ besides j itself, but there are ways of making this notion precise, although we will not discuss it in this version of the paper.

We will write the above model as Υ_∞ so that

$$\Upsilon_\infty(\mathbf{X}_0, \mathbf{Z}_0, \mathbf{U}) := (\mathbf{X}, \mathbf{Z}, \mathbf{U})$$

The geometric interpretation of Υ_∞ is similar to that of Υ_0. We begin with a Poisson collection of particles uniformly distributed in location-level space $\mathbb{R} \times \mathbb{R}^+$. Projected onto the location component, the particles may be viewed as an infinite-density collection occupying the real line. Over time, the particles perform independent, horizontal Brownian motions in the location-level space—the particles change their locations, but their levels do not change.

Each particle is assigned an initial type in the type space E: the distribution of types may depend on the particles' locations but are independent of their levels. Neutral events occur analogously to those in model Υ_0. Each particle looks down to copy the types of lower-level particles according to Poisson counting processes driven by local times.

Each particle j experiences potential selective events according to a Poisson counting process with constant rate $\bar{\sigma}$. When such an event occurs, a random particle type is chosen as some unspecified, deterministic function ψ of the empirical location/type/level measure, the location of particle j, and an independent random input η. Intuitively, we imagine that this type has been picked at random from the collection of all particles arbitrarily close to j in location space. According to a coin flip biased by the selective advantage the new type has over j's present type, j either adopts the new type or keeps its present type. Between interactions, the particles mutate independently.

3.3 Coupling to the Brownian Moran Model

The particle system Υ_0 of Section 3.1 describes a K-density collection of Brownian motions that interact locally according to an ordering determined by the particles' levels. In this section, we will show that

$$\tilde{\xi}^K =^d \xi^K$$

are equal in distribution as measure-valued processes, where $\tilde{\xi}^K$ is the empirical location/type measure-valued process for model $\tilde{\Upsilon}_0$ of Section 1.4 defined in (1.23) and ξ^K is the empirical location/type measure-valued process for model Υ_0 defined in (3.1).

We will show this by means of an intermediate ordered model with a level structure more complex than that of Υ_0. In addition, a modified version of this intermediate model will allow us to prove some important properties of the infinite-density model Υ_∞.

To construct the intermediate model, fix $K > 0$, and let the following be independent:

- $(\mathbf{X}(0), \mathbf{Z}(0))$ some indexing of the points of a Poisson point process with mean measure $K\nu_0$;
- $\{U_j : j \in \mathbb{N}\}$ iid uniform on $[0, K]$;
- $\{W_j : j \in \mathbb{N}\}$ iid standard Brownian motions;
- $\{N^\lambda_{\{i,j\}} : i \neq j \in \mathbb{N}\}$ and $\{\bar{N}^\sigma_{ij} : i \neq j \in \mathbb{N}\}$ iid rate one Poisson counting processes;
- $\{\pi_{\{i,j\},k} : i \neq j, k \in \mathbb{N}\}$ independent uniformly random permutations $\{i, j\} \leftrightarrow \{i, j\}$;
- $\{Y_{jk} : j \in \mathbb{N}, k \in \mathbb{Z}^+\}$ iid copies of Y;
- $\{\zeta_{jk} : j \in \mathbb{N}, k \in \mathbb{Z}^+\}$ iid uniform on $[0, 1]$.

Note that we have indexed $N^\lambda_{\{i,j\}}$ and $\pi_{\{i,j\},k}$ by the *unordered* particle pairs $\{i \neq j\}$.

Define the particle locations

$$\tilde{X}_j := X_j(0) + \sqrt{\theta} W_j$$

For each unordered pair $\{i \neq j\}$, let

$$L_{\{i,j\}} := L^0_t(\tilde{X}_i - \tilde{X}_j) = L^0_t(\tilde{X}_j - \tilde{X}_i)$$

be the local time at zero of $\tilde{X}_i - \tilde{X}_j$ and let $\{T^\lambda_{\{i,j\},k} : k \in \mathbb{N}\}$ be the ordered jump times of the counting process $V^\lambda_{\{i,j\}}$ defined by

$$V^\lambda_{\{i,j\}} := N^\lambda_{\{i,j\}}\left(\frac{\lambda}{2\theta} L_{\{i,j\}}\right)$$

For each ordered particle pair (i,j) with $i \neq j$, let

$$\bar{V}^\sigma_{ij} := \bar{N}^\sigma_{ij}\left(\frac{\bar{\sigma}}{2\theta} L_{\{i,j\}}\right)$$

The following theorem completes the specification of the intermediate model by defining $[0,K]$-valued level processes $\tilde{U}_j$ and $\mathbb{N}$-valued permutation processes Φ_j.

Theorem 3.3.1. *Define the filtrations*

$$\mathfrak{F}_t := \sigma\left\{T^\lambda_{\{i,j\},k}, \pi_{\{i,j\},k} : T^\lambda_{\{i,j\},k} \leq t, i \neq j, k \in \mathbb{N}\right\}$$
$$\hat{\mathfrak{F}}_t := \mathfrak{F}_t \vee \sigma\{U_j : j \in \mathbb{N}\}$$

For each j*, we may almost surely strictly order the set of ordered pairs*

$$\{(T^\lambda_{\{i,j\},k}, \pi_{\{i,j\},k}) : i \neq j, k \in \mathbb{N}\}$$

by the order of their first component. Let $\{(T^\lambda_{j,k}, \pi_{j,k})\}_k$ *be this ordering (such that* $T^\lambda_{j,1} < T^\lambda_{j,2} < \dots$*). Then,*

1. *There exist* $\{\hat{\mathfrak{F}}_{t-}\}$*-adapted,* $[0,K]$*-valued, left-continuous, jump processes* $\tilde{U}_j$ *satisfying*

$$\tilde{U}_j(t) = \begin{cases} U_j, & 0 \leq t \leq T^\lambda_{j,1}; \\ \tilde{U}_{\pi_{j,l}(j)}(T^\lambda_{j,l}), & T^\lambda_{j,l} < t \leq T^\lambda_{j,l+1},\ l \in \mathbb{N}. \end{cases}$$

and $\{\mathfrak{F}_{t-}\}$*-adapted,* $\mathbb{N}$*-valued, left-continuous, jump processes* Φ_j *satisfying*

$$\tilde{U}_{\Phi_j(t)}(t) = U_j$$

for all $t \geq 0$*. (Note that this implies* $j \mapsto \Phi_j(t)$ *is almost surely a bijection* $\mathbb{N} \leftrightarrow \mathbb{N}$ *for all* $t \in \mathbb{Q}^+$ *and so simultaneously for all* $t \in \mathbb{R}^+$*.)*

2. *For each fixed* $t \geq 0$*, conditioned on*

$$\sigma\left\{\int_0^s 1_{\{\tilde{U}_i(r)<\tilde{U}_j(r)\}} dV^\lambda_{\{i,j\}}(r), \int_0^s 1_{\{\tilde{U}_i(r)\geq\tilde{U}_j(r)\}} dV^\lambda_{\{i,j\}}(r), 0 \leq s < t\right\}$$

the $\{\tilde{U}_j(t) : j \in \mathbb{N}\}$ *are iid uniformly distributed on* $[0, K]$*. This implies that, conditioned on* $\sigma\left\{T^\lambda_{\{i,j\},k} : i \neq j, k \in \mathbb{N}\right\}$*, the indicators*

$$1_{\left\{\tilde{U}_i\left(T^\lambda_{\{i,j\},k}\right)<\tilde{U}_j\left(T^\lambda_{\{i,j\},k}\right)\right\}}$$

are iid coin flips.

We will prove Theorem 3.3.1 by means of the following lemma.

Lemma 3.3.2. *Let* $\tau \subseteq (0, \infty)$ *be a countable set of times, and let* $\sigma \colon \tau \to \mathfrak{P}_1(\mathbb{N})$ *where* $\mathfrak{P}_1(\mathbb{N})$ *is the collection of nonempty, finite subsets of* $\mathbb{N}$*. Let* $n_t := |\sigma_t|$ *for all* $t \in \tau$*. For each* $j \in \mathbb{N}$*, define* $\tau_j := \{t \in \tau : j \in \sigma_t\}$*. For each* $j \in \mathbb{N}$ *and* $T > 0$*, define* $\Sigma^T_j \colon [0, T] \to \mathfrak{P}_1(\mathbb{N})$ *by*

$$\Sigma^T_j(t) := \{j\} \cup \{i : \exists m \in \mathbb{N}, \{t_1 < t_2 < \cdots < t_m\} \subseteq \tau \cap [t, T) \text{ such that } i \in \sigma_{t_1}, j \in \sigma_{t_m}, \sigma_{t_l} \cap \sigma_{t_{l+1}} \neq \emptyset\}$$

Note that $\Sigma^T_j(t)$ *is left-continuous and monotone decreasing in* t*.*

Let $\{\pi_t : t \in \tau\}$ *be a sequence of independent, uniformly random permutations* $\pi_t(\omega) \colon \sigma_t \leftrightarrow \sigma_t$*; and let* $\{U_j : j \in \mathbb{N}\}$ *be iid, uniformly distributed on* $[0, K]$*, and independent of the* $\{\pi_t\}$*. Define the filtrations*

$$\mathfrak{F}_t := \sigma\{\pi_s : s \in \tau \cap [0, t]\}$$
$$\hat{\mathfrak{F}}_t := \mathfrak{F}_t \vee \sigma\{U_j : j \in \mathbb{N}\}$$

for all $t \in \mathbb{R}^+$.

Under the hypothesis that the sets $\tau_j \cap [0,T]$ *and* $\Sigma_j^T(0)$ *are finite for all* $j \in \mathbb{N}$ *and* $T > 0$, *we may order each* τ_j *as* $\tau_j = \{t_{j,1} < t_{j,2} < \dots\}$, *and there exist* $\{\hat{\hat{\mathfrak{F}}}_{t-}\}$*-adapted,* $[0,K]$*-valued, left-continuous, jump processes* $\tilde{U}_j$ *satisfying*

$$\tilde{U}_j(t) = \begin{cases} U_j, & 0 \le t \le t_{j,1}; \\ \tilde{U}_{\pi_{t_{j,l}}(j)}(t_{j,l}), & t_{j,l} < t \le t_{j,l+1},\ l \in \mathbb{N}. \end{cases}$$

and $\{\mathfrak{F}_{t-}\}$*-adapted,* $\mathbb{N}$*-valued, left-continuous, jump processes* Φ_j *satisfying*

$$\tilde{U}_{\Phi_j(t)}(t) = U_j$$

for all $t \ge 0$. *In addition, the* $\tilde{U}_j$ *and* Φ_j *may be taken to be deterministic (measurable) functions of* τ, σ, π_t, *and* U_j.

Furthermore, for each $t \in \tau$, *let* ρ_t *be the random ranks of* $\{\tilde{U}_j(t) : j \in \sigma_t\}$ *such that* $\rho_t(\omega) \colon \sigma_t \to \{1, \dots, n_t\}$. *Then, for all* $T > 0$, $\{\tilde{U}_j(T)\}_{j \in \mathbb{N}}$ *conditioned on* $\sigma\{\rho_t : t \in \tau \cap [0,T)\}$ *are iid, uniformly distributed on* $[0,K]$. *In particular,* $\{\rho_t : t \in \tau\}$ *are mutually independent, each uniformly random over the permutations* $\sigma_t \leftrightarrow \{1, 2, \dots, n_t\}$.

Proof. Fix $T > 0$, $\sigma' \in \mathfrak{P}_1(\mathbb{N})$, and finite $\tau' \subseteq \tau \cap [0,T)$.

Define

$$\sigma_\star := \sigma' \cup (\cup_{t \in \tau'} \sigma_t)$$

and

$$\Sigma_{\sigma_\star}^T(t) := \cup_{j \in \sigma_\star} \Sigma_j^T(t)$$

Note that $\sigma_\star$ is finite, so $\Sigma_{\sigma_\star}^T(0)$ is finite with $\Sigma_{\sigma_\star}^T(t)$ monotone decreasing in t. Write $\Sigma_\star = \Sigma_{\sigma_\star}^T$.

Let us define a set of times

$$\tau_\star := \{t \in \tau \cap [0,T) : \sigma_t \cap \Sigma_\star(t) \neq \emptyset\} = \{t \in \tau \cap [0,T) : \sigma_t \subseteq \Sigma_\star(t)\}$$

Note that

$$\tau_\star \subseteq \cup_{j \in \Sigma_\star(0)} (\tau_j \cap [0,T])$$

and thus $\tau_\star$ is finite. Therefore, we may order it as

$$\tau_\star = \{t_1 < t_2 < \cdots < t_m\}$$

Define $t_{m+1} = T$.

Now, working inductively for $l = 1, \ldots, m+1$, we will define

- $[0, K]$-valued random variables $\tilde{U}_j(t_l)$ for all $j \in \Sigma_\star(t_l)$; and
- $\Sigma_\star(t_l)$-valued random variables $\Phi_j(t_l)$ for all $j \in \Sigma'(t_l)$ where $\Sigma'(t_l) := \{j : \exists j' \in \Sigma_\star(t_l),\ \tilde{U}_{j'}(t_l) = U_j\} \subseteq \Sigma_\star(0)$

satisfying the inductive hypothesis:

- $\tilde{U}_{\Phi_j(t_l)}(t_l) = U_j$ for all $j \in \Sigma'(t_l)$;
- $\tilde{U}_j(t_l)$ are iid uniformly distributed on $[0, K]$ and $\hat{\hat{\mathfrak{F}}}_{t_l-}$-measurable;
- $\Phi_j(t_l)$ are $\mathfrak{F}_{t_l-}$-measurable;
- the random ranks ρ_{t_k} of $\left\{\tilde{U}_j(t_k) : j \in \sigma_{t_k} \subseteq \Sigma_\star(t_k)\right\}$ are iid uniformly random permutations $\sigma_t \leftrightarrow \{1, \ldots, n_{t_k}\}$ for all $k \leq l-1$;
- $\left\{\tilde{U}_j(t_l) : j \in \Sigma_\star(t_l)\right\}$ and $\{\rho_{t_k} : k = 1, \ldots, l-1\}$ are mutually independent.

Initially, define $\tilde{U}_j(t_1) := U_j$ and $\Phi_j(t_1) := j$ for all $j \in \Sigma_\star(t_1)$. Then, the inductive hypothesis is satisfied for $l = 1$.

Suppose the hypothesis is satisfied for l. For all $j \in \Sigma_\star(t_{l+1})$, define

$$\tilde{U}_j(t_{l+1}) := \begin{cases} \tilde{U}_{\pi_{t_l}(j)}(t_l), & j \in \Sigma_\star(t_{l+1}) \cap \sigma_{t_l}; \\ \tilde{U}_j(t_l), & j \in \Sigma_\star(t_{l+1}) \setminus \sigma_{t_l}. \end{cases}$$

and for all $j \in \Sigma'(t_{l+1})$ define

$$\Phi_j(t_{l+1}) := \begin{cases} \pi_{t_l}^{-1}(\Phi_j(t_l)), & \Phi_j(t_l) \in \sigma_{t_l}; \\ \Phi_j(t_l), & \Phi_j(t_l) \notin \sigma_{t_l}. \end{cases}$$

Note we have $\Phi_j(t_{l+1}) \in \Sigma_\star(t_{l+1})$ and $\tilde{U}_{\Phi_j(t_{l+1})}(t_{l+1}) = U_j$: for, if $\Phi_j(t_l) \notin \sigma_{t_l}$, then

$$\Phi_j(t_{l+1}) = \Phi_j(t_l) \in \Sigma_\star(t_l) \setminus \sigma_{t_l} \subseteq \Sigma_\star(t_{l+1}) \setminus \sigma_{t_l}$$

implying

$$\tilde{U}_{\Phi_j(t_{l+1})}(t_{l+1}) = \tilde{U}_{\Phi_j(t_l)}(t_l) = U_j$$

while if $\Phi_j(t_l) \in \sigma_{t_l}$, then $j \in \Sigma'(t_{l+1})$ implies we have $j' \in \Sigma_\star(t_{l+1})$ with $\tilde{U}_{j'}(t_{l+1}) = U_j$. As $\tilde{U}_{\Phi_j(t_l)}(t_l) = U_j$, the uniqueness of the U_j implies $j' \in \sigma_{t_l}$ with $\pi_{t_l}(j') = \Phi_j(t_l)$. But then,

$$\Phi_j(t_{l+1}) = \pi_{t_l}^{-1}(\Phi_j(t_l)) = j' \in \Sigma_\star(t_{l+1}) \cap \sigma_{t_l}$$

That, in turn, implies

$$\tilde{U}_{\Phi_j(t_{l+1})}(t_l) = \tilde{U}_{\pi_{t_l}(j')}(t_l) = \tilde{U}_{\Phi_j(t_l)}(t_l) = U_j$$

Noting that ρ_{t_l} is measurable with respect to $\sigma\{\tilde{U}_j(t_l) : j \in \sigma_{t_l}\}$, we have

$$\begin{aligned}
&E\left[\prod_{j\in\Sigma_\star(t_{l+1})} h_j(\tilde{U}_j(t_{l+1})) \prod_{k=1}^{l} g_k(\rho_{t_k})\right] \\
&= E\left[\prod_{j\in\Sigma_\star(t_{l+1})\setminus\sigma_{t_l}} h_j(\tilde{U}_j(t_l)) \prod_{j\in\Sigma_\star(t_{l+1})\cap\sigma_{t_l}} h_j(\tilde{U}_{\pi_{t_l}(j)}(t_l)) \prod_{k=1}^{l} g_k(\rho_{t_k})\right] \\
&= E\left[\prod_{k=1}^{l-1} g_k(\rho_{t_k})\right] E\left[\prod_{j\in\Sigma_\star(t_{l+1})\setminus\sigma_{t_l}} h_j(\tilde{U}_j(t_l))\right] E\left[g_l(\rho_{t_l}) \prod_{j\in\Sigma_\star(t_{l+1})\cap\sigma_{t_l}} h_j(\tilde{U}_{\pi_{t_l}(j)}(t_l))\right] \\
&= E\left[\prod_{k=1}^{l-1} g_k(\rho_{t_k})\right] E\left[\prod_{j\in\Sigma_\star(t_{l+1})\setminus\sigma_{t_l}} h_j(\tilde{U}_j(t_l))\right] E\left[g_l(\rho_{t_l})\right] \\
&\qquad\qquad \cdot E\left[\prod_{j\in\Sigma_\star(t_{l+1})\cap\sigma_{t_l}} h_j(\tilde{U}_j(t_l))\right]
\end{aligned}$$

where the second equality follows from the inductive hypothesis—giving the independence of $\sigma\{\rho_{t_k} : k = 1, \ldots, l-1\}$, $\sigma\{\tilde{U}_j(t_l) : j \in \Sigma_\star(t_{l+1}) \setminus \sigma_{t_l}\}$, and $\sigma\{\tilde{U}_j(t_l) : j \in \Sigma_\star(t_{l+1}) \cap \sigma_{t_l}\} \vee \sigma\{\pi_{t_l}\}$—and the third equality follows from the fact that π_{t_l} is a uniformly random permutation independent of $\sigma\{\tilde{U}_j(t_l) : j \in \Sigma_\star(t_{l+1})\}$.

The fact that the $\tilde{U}_j(t_l)$ for $j \in \Sigma_\star(t_l)$ are iid, uniformly distributed on $[0, K]$ then gives us the inductive hypothesis for $l+1$.

Therefore, the inductive hypothesis holds for $m+1$. In particular, $\{\tilde{U}_j(T) : j \in \sigma' \subseteq \Sigma_\star(T)\}$ and $\{\rho_t : t \in \tau' \subseteq \tau_\star\}$ are mutually independent with $\tilde{U}_j(T)$ uniformly distributed on $[0, K]$ and ρ_t uniformly random permutations of $\{1, \dots, n_t\}$.

Finally, note that the definitions of $\tilde{U}_j(t)$ and $\Phi_j(t)$, for *any* $j \in \mathbb{N}$ and $t \in \tau$, that arise by the inductive process described above are unique, irrespective of the initial choice of T, σ', and τ'. Moreover, these random variables *can* be defined for all such j and t by taking $T = t$, $\sigma' = \{j\}$, and $\tau' = \{t\}$. Thus, we can uniquely define processes $\tilde{U}_j$ and Φ_j for all $j \in \mathbb{N}$ by

$$\tilde{U}_j(t) := \begin{cases} \tilde{U}_j(t_1), & 0 \le t \le t_1; \\ \tilde{U}_j(t_l), & t_l < t \le t_{l+1},\ l \in \mathbb{N}. \end{cases}$$

and analogously

$$\Phi_j(t) := \begin{cases} \Phi_j(t_1), & 0 \le t \le t_1; \\ \Phi_j(t_l), & t_l < t \le t_{l+1},\ l \in \mathbb{N}. \end{cases}$$

These processes are easily seen to satisfy the conditions of the lemma. □

Proof of Theorem 3.3.1. We may take our probability space to be a product space $(\Omega_1 \times \Omega_2, \mathfrak{F}_1 \times \mathfrak{F}_2, P_1 \times P_2)$ such that $(\mathbf{X}(0), \mathbf{Z}(0))$, W_j, $N^\lambda_{\{i,j\}}$, $\bar{N}^\sigma_{ij}$, Y_{jk}, and ζ_{jk} are $\mathfrak{F}_1 \times \Omega_2$-measurable while U_j and $\pi_{\{i,j\},k}$ are $\Omega_1 \times \mathfrak{F}_2$-measurable.

Define $\tau := \{T^\lambda_{\{i,j\},k} : i \neq j, k \in \mathbb{N}\}$, and let n_t, τ_i, and Σ^T_i be defined as in Lemma 3.3.2. Almost surely P_1, the $T^\lambda_{\{i,j\},k}$ are all distinct, and we may unambiguously define $\sigma \colon \tau \to \mathfrak{P}_1(\mathbb{N})$ by $\sigma(T^\lambda_{\{i,j\},k}) = \{i, j\}$. Moreover, we will establish in Lemma 3.3.4 that, almost surely P_1, we have $\tau_j \cap [0, T]$ and $\Sigma^T_j(0)$ finite for all $j \in \mathbb{N}$ and $T > 0$. For all such $\omega_1 \in \Omega_1$, we may apply Lemma 3.3.2 to produce processes $\tilde{U}_j(\omega_1)$ and $\Phi_j(\omega_1)$ defined on the probability space $(\Omega_2, \mathfrak{F}_2, P_2)$.

As the $\tilde{U}_j$ and Φ_j are measurable functions of random variables measurable with respect to the product space, the $\tilde{U}_j$ and Φ_j are actually random variables on this larger space. They are easily seen to satisfy the statement of Theorem 3.3.1. $\square$

The proof of Theorem 3.3.1 relies on the following two lemmas which establish that, almost surely, $\tau_j \cap [0, T]$ and $\Sigma_j^T(0)$ are finite for all $j \in \mathbb{N}$ and $T > 0$.

Lemma 3.3.3. *Let* $\Gamma := \left\{(t, x) \in \mathbb{R}^+ \times \mathbb{R} : \exists j \in \mathbb{N},\ \tilde{X}_j(t) = x\right\}$ *be the graph of the Brownian paths, and let* $\Gamma_t := \pi(\Gamma \cap ([0, t] \times \mathbb{R}))$ *(with* $\pi\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *the projection map) be the set of points covered by the paths up to time* t*. Then, for any fixed* $t \geq 0$*, we have*

$$P(m \notin \Gamma_t \text{ i.o. } m \in \mathbb{N}) = P(m \notin \Gamma_t \text{ i.o. } m \in -\mathbb{N}) = 1$$

That is, almost surely, infinitely many positive and negative integers are not *hit by any of the Brownian motions up to time* t*.*

Proof. Let $\tilde{X}_j[0, t] := \{\tilde{X}_j(s) : 0 \leq s \leq t\}$ be the interval covered by $\tilde{X}_j$ up to time t. For each $m \in \mathbb{Z}$, define

$$N_m := \sum_j 1_{\{m \in \tilde{X}_j[0,t]\}}$$

Note that

$$(\tilde{X}_j(0), W_j) \sim \textit{Poisson}(K\ell_{\mathbb{R}} \times \mathfrak{W})$$

for $\mathfrak{W}$ the law of W_1, so as we have

$$N_m = \sum_j 1_{\left\{(\tilde{X}_j(0), W_j) \in \left\{(x,w) : m \in \{x + \sqrt{\theta} w_s : 0 \leq s \leq t\}\right\}\right\}}$$

it follows that N_m is Poisson with mean

$$\begin{aligned}
&(K\ell_{\mathbb{R}} \times \mathfrak{W}) \left(\left\{(x, w) : m \in \left\{x + \sqrt{\theta} w_s : 0 \leq s \leq t\right\}\right\}\right) \\
&= K \int P\left(m \in x + \sqrt{\theta} W_s : 0 \leq s \leq t\right) dx \\
&= 2K \sqrt{\frac{2}{\pi \theta t}}
\end{aligned}$$

Noting that the ergodicity of the original system over location space $\mathbb{R}$ ensures the ergodicity of the stationary sequence $\{N_m : m \in \mathbb{Z}\}$, the ergodic theorem implies

$$\frac{1}{M}\sum_{m=1}^{M} 1_{\{N_m=0\}} \underset{M\to\infty}{\longrightarrow} P(N_1 = 0) = e^{-2K\sqrt{\frac{2}{\pi\theta t}}} > 0$$

In particular, this implies

$$P(N_m = 0 \text{ i.o. } m \in \mathbb{N}) = 1$$

and similarly $P(N_m = 0 \text{ i.o. } m \in -\mathbb{N}) = 1$, giving the result. □

Lemma 3.3.4. *Almost surely, we have* $\tau_j \cap [0,T]$ *and* $\Sigma_j^T(0)$ *finite for all* $j \in \mathbb{N}$ *and* $T > 0$.

Remark 3.3.1. τ_j is the set of neutral lookdowns in which j participates, either as parent or child. $\Sigma_j^T(0)$ is the *influence set* of j at time T. In the neutral case, it is the set of all particles whose type might have "influenced" j's type at times prior to T (through a chain of reproduction events). In particular, if we take a realization of the process and change the initial type of a single particle i without changing the genealogy, a particle j's type immediately before time T may only change if $i \in \Sigma_j^T(0)$—if i is in j's influence set.

Thus, this lemma merely states that, up to a fixed time $T > 0$, every particle j has experienced only finitely many neutral reproduction events and has been influenced (through chains of neutral events) by the types of only a finite number of particles.

Proof. Let the notation be as in the previous lemma. As Γ_0 is a Poisson point process on $\mathbb{R}$, we have $|\Gamma_0 \cap [m_1, m_2]| < \infty$ for all $m_1 < m_2 \in \mathbb{Z}$ almost surely.

Fix $j \in \mathbb{N}$ and $T \in \mathbb{N}$. Almost surely, there exists some (random) $M \in \mathbb{N}$ such that $\tilde{X}_j(0) \in (-M, M)$. By the previous lemma, there almost surely exists some random integer $M_1 \leq -M$ such that $M_1 \notin \Gamma_T$. Similarly, there almost surely exists some random integer $M_2 \geq M$ such that $M_2 \notin \Gamma_T$. Therefore,

$$\Sigma_j^T(0) \subseteq \{i : \tilde{X}_i[0,t] \subset [M_1, M_2]\} = \{i : \tilde{X}_i(0) \in [M_1, M_2]\}$$

and $|\Sigma_j^T(0)| \leq |\Gamma_0 \cap [M_1, M_2]| < \infty$ almost surely.

Also, as $V^\lambda_{\{i,j\}}(T) = 0$ for all $i \notin \Sigma_j^T(0)$, we have

$$|\tau_j \cap [0,T]| = \sum_{\substack{i \in \Sigma_j^T(0) \\ i \neq j}} N^\lambda_{\{i,j\}}\left(\frac{\lambda}{2\theta} L_{\{i,j\}}(T)\right)$$

Since $L_{\{i,j\}}(T) < \infty$ for all $i \neq j$ almost surely and as the sum is over an almost surely finite set, it follows that $|\tau_j \cap [0,T]| < \infty$ almost surely.

As there are countably many particles, we have $|\Sigma_j^T(0)|$ and $|\tau_j \cap [0,T]|$ finite for all j at all integral time points T almost surely. Since these cardinalities are monotone increasing in T, they are finite for *all* $T > 0$ as well. □

Having completed the specification of the intermediate model, we may now turn to the task of coupling the ordered model Υ_0 of Section 3.1 to the symmetric model $\tilde{\Upsilon}_0$. To do this, we will use the intermediate model to construct two new particle models. The first, a symmetric model, will be "constructed" by essentially ignoring the level structure of the intermediate model. The second, an ordered model, will be constructed by ignoring the indexing of the intermediate model and taking the "particles" to be *defined* by the level structure $\tilde{\mathbf{U}} = (\tilde{U}_j)_j$. That is, a "particle" in our ordered model will have a fixed level u, and its location and type at any time t will be given by the location and type of whichever real, indexed particle in the intermediate model happens to have level u at time t.

To construct the symmetric model, we define jump processes $\tilde{V}^\lambda_{ij}$ by

$$\tilde{V}^\lambda_{ij}(t) := \int_0^t 1_{\{\tilde{U}_i(s) < \tilde{U}_j(s)\}} dV^\lambda_{\{i,j\}}(s)$$

By Theorem 3.3.1, conditioned on all $T^\lambda_{\{i,j\},k}$, the $1_{\{\tilde{U}_i(T^\lambda_{\{i,j\},k}) < \tilde{U}_j(T^\lambda_{\{i,j\},k})\}}$ are iid fair coin flips. Therefore, we have

$$\tilde{V}^\lambda_{ij} =^d N^\lambda_{ij}\left(\frac{\lambda}{4\theta} L_{\{i,j\}}\right) \tag{3.8}$$

for iid, rate one Poisson counting processes N^{λ}_{ij}.

Let us define the type process $\tilde{\mathbf{Z}}$ in the notation of Remark 1.3.2 by

$$(\tilde{\mathbf{Z}}, \tilde{\mathbf{V}}, \tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\gamma}}, \tilde{\mathfrak{i}}) := \mathfrak{T}(\mathbf{Z}(0), \mathbf{Y}, \tilde{\mathbf{V}}^{\lambda}, \bar{\mathbf{V}}^{\sigma}, \zeta, \bar{\sigma}^{-1}\sigma)$$

A unique solution $\tilde{\mathbf{Z}}$ exists: this can be established in much the same way we established the existence of $\boldsymbol{\Phi}$ and $\tilde{\mathbf{U}}$. Moreover, by (3.8), we have

$$(\mathbf{X}, \tilde{\mathbf{Z}}) =^{d} \tilde{\Upsilon}_0(\mathbf{X}(0), \mathbf{Z}(0))$$

(where we have taken $\lambda = \lambda_0$, $\sigma = \sigma_0$, and $\bar{\sigma} = \bar{\sigma}_0$).

To construct the ordered model, define the process

$$X_j(t) := \tilde{X}_{\Phi_j(t)}(t)$$

Noting that this process is continuous, we may write

$$X_j(t) = X_j(0) + \sqrt{\theta} \sum_{j' \in \mathbb{N}} \int_0^t 1_{\{\Phi_j(s)=j'\}} dW_{j'}(s)$$

and so X_j are continuous martingales satisfying

$$\begin{aligned} [X_i, X_j]_t &= \theta \sum_{j' \in \mathbb{N}} \int_0^t 1_{\{\Phi_i(s)=j'\}} 1_{\{\Phi_j(s)=j'\}} ds \\ &= \theta t 1_{\{i=j\}} \end{aligned}$$

Therefore, by Lévy's characterization, we have

$$X_j =^{d} X_j(0) + \sqrt{\theta} W_j$$

Define counting processes V^{λ}_{ij} by

$$V^{\lambda}_{ij}(t) := 1_{\{U_i < U_j\}} \sum_{i', j' \in \mathbb{N}} \int_0^t 1_{\{\Phi_i(s)=i'\}} 1_{\{\Phi_j(s)=j'\}} dV^{\lambda}_{\{i', j'\}}(s)$$

and counting processes $\bar{V}^{\sigma,\Phi}_{ij}$ by

$$\bar{V}^{\sigma,\Phi}_{ij}(t) := \sum_{i', j' \in \mathbb{N}} \int_0^t 1_{\{\Phi_i(s)=i'\}} 1_{\{\Phi_j(s)=j'\}} d\bar{V}^{\sigma}_{i', j'}(s)$$

Writing $\tilde{X}_{i',j'} := \tilde{X}_{i'} - \tilde{X}_{j'}$ and $X_{ij} := X_i - X_j$, by the Meyer-Tanaka formula, we have for $i' \neq j'$ that

$$|\tilde{X}_{i',j'}(t)| = |\tilde{X}_{i',j'}(0)| + \int_0^t \operatorname{sgn}(\tilde{X}_{i',j'}(s-))d\tilde{X}_{i',j'}(s) + L_{\{i',j'\}}(t)$$

However, for $i \neq j$, we have

$$\begin{aligned}
|X_{ij}(t)| - |X_{ij}(0)| &= |\tilde{X}_{\Phi_i(t),\Phi_j(t)}(t)| - |\tilde{X}_{i,j}(0)| \\
&= \sum_{i',j' \in \mathbb{N}} \int_0^t 1_{\{\Phi_i(s)=i'\}} 1_{\{\Phi_j(s)=j'\}} d|\tilde{X}_{i',j'}|(s) \\
&= \sum_{i',j' \in \mathbb{N}} \int_0^t 1_{\{\Phi_i(s)=i'\}} 1_{\{\Phi_j(s)=j'\}} \operatorname{sgn}(\tilde{X}_{i',j'}(s-))d\tilde{X}_{i',j'}(s) \\
&\quad + \sum_{i',j' \in \mathbb{N}} \int_0^t 1_{\{\Phi_i(s)=i'\}} 1_{\{\Phi_j(s)=j'\}} dL_{\{i',j'\}}(s)
\end{aligned}$$

Yet,

$$\begin{aligned}
&\int_0^t \operatorname{sgn}(X_{ij}(s-))dX_{ij}(s) \\
&\qquad = \sum_{i',j' \in \mathbb{N}} \int_0^t 1_{\{\Phi_i(s)=i'\}} 1_{\{\Phi_j(s)=j'\}} \operatorname{sgn}(\tilde{X}_{i',j'}(s-))d\tilde{X}_{i',j'}(s)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|X_{ij}(t)| - |X_{ij}(0)| &= \int_0^t \operatorname{sgn}(X_{ij}(s-)dX_{ij}(s) \\
&\quad + \sum_{i',j' \in \mathbb{N}} \int_0^t 1_{\{\Phi_i(s)=i'\}} 1_{\{\Phi_j(s)=j'\}} dL_{\{i',j'\}}(s)
\end{aligned}$$

and, again by the Meyer-Tanaka formula, it follows that the local time at zero of $X_i - X_j$ is given by

$$L_{ij}(t) := L_t^0(X_i - X_j) = \sum_{i',j' \in \mathbb{N}} \int_0^t 1_{\{\Phi_i(s)=i'\}} 1_{\{\Phi_j(s)=j'\}} dL_{\{i',j'\}}(s)$$

Now, for $U_i < U_j$, we have

$$V_{ij}^{\lambda}(t) - \frac{\lambda}{2\theta}L_{ij} = \sum_{i',j' \in \mathbb{N}} \int_0^t 1_{\{\Phi_i(s)=i'\}} 1_{\{\Phi_j(s)=j'\}} d\left(V_{\{i',j'\}}^{\lambda} - \frac{\lambda}{2\theta}L_{\{i',j'\}}\right)(s)$$

a martingale. In fact, it is a martingale with respect to

$$\mathfrak{F}_t^{V_{ij}^{\lambda}} \vee \mathfrak{F}_{\infty}^{L_{ij}}$$

and so

$$V_{ij}^{\lambda} =^d N_{ij}^{\lambda}\left(\frac{\lambda}{2\theta}L_{ij}\right) \tag{3.9}$$

for iid, rate one Poisson counting processes N_{ij}^{λ}. Similarly,

$$\bar{V}_{ij}^{\sigma,\Phi} =^d \bar{N}_{ij}^{\sigma}\left(\frac{\bar{\sigma}}{2\theta}L_{ij}\right) \tag{3.10}$$

Define the type process $\mathbf{Z}$ by

$$Z_j(t) := \tilde{Z}_{\Phi_j(t)}(t)$$

Lemma 3.3.5. *If we let*

$$(\mathbf{Z}', \mathbf{V}, \boldsymbol{\tau}, \boldsymbol{\gamma}, \boldsymbol{\iota}) := \mathfrak{T}(\mathbf{Z}(0), \mathbf{Y}, \mathbf{V}^{\lambda}, \bar{\mathbf{V}}^{\sigma,\Phi}, \boldsymbol{\zeta}, \bar{\sigma}^{-1}\sigma)$$

then we have

$$(\mathbf{X}, \mathbf{Z}, \mathbf{U}) =^d (\mathbf{X}, \mathbf{Z}', \mathbf{U}) =^d \Upsilon_0(\mathbf{X}(0), \mathbf{Z}(0), \mathbf{U})$$

Proof. Observe that the neutral interactions play identical roles with respect to the processes $\mathbf{Z}$ and $\mathbf{Z}'$: when V_{ij}^{λ} jumps, Z_j and Z_j' are "set" to Z_i and Z_i' respectively.

Similarly, for selective events, when $\bar{V}_{ij}^{\sigma,\Phi}$ jumps, the types Z_j and Z_j' are potentially "set" to Z_i and Z_i' respectively.

Since the genealogical structures are identical, it follows by the strong Markov property of the Y_{jk} (with respect to reproduction event times) that

$$(\mathbf{X}, \mathbf{Z}, \mathbf{U}) =^d (\mathbf{X}, \mathbf{Z}', \mathbf{U})$$

Equations (3.9) and (3.10) imply

$$(\mathbf{X}, \mathbf{Z}', \mathbf{U}) =^{d} \Upsilon_0(\mathbf{X}(0), \mathbf{Z}(0), \mathbf{U})$$

giving the result. □

This is sufficient to establish the following proposition.

Proposition 3.3.6. *We have*

$$\tilde{\xi}^K =^{d} \xi^K$$

for $\tilde{\xi}^K$ *defined in* (1.23) *and* ξ *defined in* (3.6).

Proof. By our derivation of the symmetric model $(\tilde{\mathbf{X}}, \tilde{\mathbf{Z}})$, we see that we have

$$(\tilde{\mathbf{X}}, \tilde{\mathbf{Z}}) =^{d} \tilde{\Upsilon}_0(Poisson(K\nu_0)) \tag{3.11}$$

which has empirical location/type measure $\tilde{\xi}^K$

Similarly, by the previous lemma, we have

$$(\mathbf{X}, \mathbf{Z}, \mathbf{U}) =^{d} \Upsilon_0(Poisson(\nu_0 \times \ell_{[0,K]})) \tag{3.12}$$

By the intermediate model coupling, the symmetric model $(\tilde{\mathbf{X}}, \tilde{\mathbf{Z}}, \tilde{\mathbf{U}})$ and the ordered model $(\mathbf{X}, \mathbf{Z}, \mathbf{U})$ models have the same empirical location/type/level (and, so in particular, location/type) process distribution, and the result follows. □

We now wish to prove a corollary of Theorem 3.3.1 that will prove useful in later sections. To do this, we must construct a "hybrid" model. From level K down, it will be similar to the intermediate model constructed in this section (but with a different selective event structure). From level K up, particles will act much as in the ordered model Υ_∞ of Section 3.2. For the purposes of neutral events, they will "look down" at lower-level particles (both above and below level K) and copy their types. To simplify notation, we will index the (intermediate model) particles below level K by the natural numbers $\mathbb{N}$ and the (ordered model) particles above level K by an isomorphic copy $\mathbb{N}' \cong \mathbb{N}$ of the natural numbers disjoint from them ($\mathbb{N}' \cap \mathbb{N} = \emptyset$). That is, we will index particles by the disjoint union $\mathbb{N} \uplus \mathbb{N}'$.

For a fixed $K > 0$, let the intermediate model be defined as above. Let the following be independent of the components of the intermediate model and each other

- $\{(X_j(0), Z_j(0), U_j) : j \in \mathbb{N}'\}$ (some indexing of the points of) a Poisson point process on $\mathbb{R} \times E \times (K, \infty)$ with mean measure $\nu_0 \times \ell_{(K,\infty)}$;
- $\{W_j : j \in \mathbb{N}'\}$ iid standard BMs;
- $\{N^\lambda_{ij} : i \in \mathbb{N} \uplus \mathbb{N}', j \in \mathbb{N}'\}$ iid rate one Poisson counting processes;
- $\{\bar{N}^\sigma_j : j \in \mathbb{N} \uplus \mathbb{N}'\}$ iid rate one Poisson counting processes;
- $\{Y_{jk} : j \in \mathbb{N}', k \in \mathbb{Z}^+\}$ iid copies of Y;
- $\{\zeta_{jk} : j \in \mathbb{N}', k \in \mathbb{Z}^+\}$ iid uniform on $[0, 1]$;
- $\{\eta_{jk} : j \in \mathbb{N} \uplus \mathbb{N}', k \in \mathbb{Z}^+\}$ iid uniform on $[0, 1]$.

For $j \in \mathbb{N}'$, let us define

$$X_j := X_j(0) + \sqrt{\theta} W_j$$

and for $i \in \mathbb{N} \uplus \mathbb{N}'$ and $j \in \mathbb{N}'$ with $i \neq j$, let us define $L_{ij} := L^0_t(X_i - X_j)$ and

$$V^\lambda_{ij} := 1_{\{U_i < U_j\}} N^\lambda_{ij}\left(\frac{\lambda}{2\theta} L_{ij}\right)$$

Finally, take the hybrid model type process $\hat{Z} = (\hat{Z}_j : j \in \mathbb{N} \uplus \mathbb{N}')$ to be given, in the notation of Remark 3.2.2 by

$$(\hat{Z}, \hat{\xi}, \hat{V}, \hat{\tau}, \hat{\gamma}, \hat{\iota}, \hat{\psi}) := \mathfrak{T}'(Z(0), \hat{X}, U, Y, \hat{V}^\lambda, \hat{V}^\lambda, \bar{V}^\sigma, \zeta, \eta, \bar{\sigma}^{-1}\sigma)$$

where the indices i and j are taken over $\mathbb{N} \uplus \mathbb{N}'$; where $\hat{X}$ is given by

$$\hat{X}_j := \begin{cases} \tilde{X}_j, & j \in \mathbb{N}; \\ X_j, & j \in \mathbb{N}'. \end{cases}$$

where $\hat{\mathbf{V}}^\lambda$ is given by

$$\hat{V}^\lambda_{ij} := \begin{cases} \tilde{V}^\lambda_{ij}, & i,j \in \mathbb{N}; \\ V^\lambda_{ij}, & i \in \mathbb{N} \uplus \mathbb{N}', j \in \mathbb{N}'; \\ 0, & i \in \mathbb{N}', j \in \mathbb{N}. \end{cases}$$

for $\tilde{V}^\lambda_{ij}$ as defined above; and where $\bar{\mathbf{V}}^\sigma = (\bar{V}^\sigma_j : j \in \mathbb{N} \uplus \mathbb{N}')$ for

$$\bar{V}^\sigma_j(t) := \bar{N}^\sigma_j(\bar{\sigma} t)$$

Take the ordered type process $\mathbf{Z} = (Z_j : j \in \mathbb{N} \uplus \mathbb{N}')$ to be given by

$$Z_j := \begin{cases} \hat{Z}_{\Phi_j(\cdot)}(\cdot), & j \in \mathbb{N}; \\ \hat{Z}_j, & j \in \mathbb{N}'. \end{cases}$$

This construction gives the following result.

Corollary 3.3.7. *Let ξ. be the empirical location/type/level process associated with the infinite-density model $\Upsilon_\infty(Poisson(\nu_0 \times \ell_{\mathbb{R}^+}))$. Then, we have the following decomposition*

$$\begin{aligned} \xi_\cdot &=^d \sum_{j \in \mathbb{N}} \delta_{(X_j(\cdot), Z_j(\cdot), U_j)} + \sum_{j \in \mathbb{N}'} \delta_{(X_j(\cdot), Z_j(\cdot), U_j)} \\ &=^{a.s.} \sum_{j \in \mathbb{N}} \delta_{(\tilde{X}_j(\cdot), \tilde{Z}_j(\cdot), \tilde{U}_j(\cdot))} + \sum_{j \in \mathbb{N}'} \delta_{(X_j(\cdot), Z_j(\cdot), U_j)} \end{aligned}$$

in the notation of the present section.

Proof. The above construction of the hybrid model gives the first equality in law. The almost sure equality follows by the intermediate model coupling. □

We will see some applications of this result in the following section.

3.4 Measure-Valued Diffusion Limit

Again, let us consider the model Υ_∞ of Section 3.2. Let $(\mathbf{X}_0, \mathbf{Z}_0, \mathbf{U})$ be *Poisson*$(\nu_0 \times \ell_{\mathbb{R}^+})$, let $(\mathbf{X}, \mathbf{Z}, \mathbf{U}) = \Upsilon_\infty(\mathbf{X}_0, \mathbf{Z}_0, \mathbf{U})$, and let ξ be the location/type/level empirical process given by

$$\xi_t := \sum_j \delta_{(X_j(t), Z_j(t), U_j)}$$

We would like to characterize the location/type distribution of particles in this infinite-density model.

To be more precise, for each $K > 0$, we may define

$$u_t^K := \sum_{U_j \le K} \delta_{\left(X_j(t), Z_j(t)\right)}$$

so that we may write, for any location/type test function $h \colon \mathbb{R} \times E \to \mathbb{R}$ the sum

$$\left\langle u_t^K, h \right\rangle := \sum_{U_j \le K} h\left(X_j(t), Z_j(t)\right)$$

If h has compact support in location space, so that there exists $\alpha > 0$ with $h(y, \cdot) \equiv 0$ for $|y| > \alpha$, then the summation is almost surely over a finite number of terms.

Now, the expressions $\frac{1}{K}\left\langle u_t^K, h \right\rangle$ characterize the average location/type distribution over all particles up to level K. We would like to study these expressions in the limit as $K \to \infty$.

Define the filtration

$$\mathfrak{F}_t^K := \sigma\left\{ \xi\big|_{\mathbb{R}\times E \times (K,\infty)}(r), \sum\nolimits_{U_j \le K} \delta_{(X_j(r), Z_j(r))}, r \le t \right\}$$

That is, the filtration $\mathfrak{F}^K$ is generated by the locations, types, and levels of the particles above level K but only the locations and types (without regard to specific level) of the particles below level K. Define the filtration $\mathfrak{F}_t^\infty := \cap_{K=1}^\infty \mathfrak{F}_t^K$.

We will assume the selection function ψ of (3.5) satisfies the following condition.

Hypothesis 3.4.1. Let ψ be such that for all $K > 0$, we have

$$\psi\Big(\Big(\sum_j \delta_{(x_j,z_j,u_j)}\Big), x_0, \eta\Big) = \psi\Big(\Big(\sum_{u_j \le K} \delta_{(x_j,z_j,\hat{u}_j)} + \sum_{u_j > K} \delta_{(x_j,z_j,u_j)}\Big), x_0, \eta\Big)$$

for all $\boldsymbol{x} \in \mathbb{R}^\infty$, $\boldsymbol{z} \in E^\infty$, $\mathbf{u} \in (\mathbb{R}^+)^\infty$, $\hat{\mathbf{u}} \in [0,K]^\infty$, $x_0 \in \mathbb{R}$, and $\eta \in [0,1]$.

We will require two lemmas, the first a version of the Martingale Backwards Convergence Theorem and the second a consequence of the coupling of the previous section.

Lemma 3.4.2. *If* $\mathrm{E}\,|X|^p < \infty$ *for some* $p \ge 1$*, then*

$$\lim_{K\to\infty} \mathrm{E}\left[X \mid \mathfrak{F}_t^K\right] = \mathrm{E}\left[X \mid \mathfrak{F}_t^\infty\right]$$

where the limit is almost sure and in L_p*.*

Proof. For $p = 1$, this is Theorem 4.6.3 of [4]. For $p > 1$, let $X_K := \mathrm{E}[X|\mathfrak{F}_t^K]$, and note $|X_K|^p \xrightarrow{a.s.} |X_\infty|^p$ by application of the theorem for the case $p = 1$. However, we have

$$|X_K|^p = \left|\mathrm{E}\left[X \mid \mathfrak{F}_t^K\right]\right|^p \le \mathrm{E}\left[|X|^p \mid \mathfrak{F}_t^K\right]$$

and the right-hand side are uniformly integrable over all K by [4, Theorem 4.5.1]. Therefore, $X_K \xrightarrow{L^p} X_\infty$. □

Lemma 3.4.3. *For* $t \ge 0$ *and* $K > 0$*, we have the conditional equality in law*

$$\mathfrak{L}\left[\sum_{U_j \le K} \delta_{(X_j(t),Z_j(t),U_j)} \,\middle|\, \mathfrak{F}_t^K\right] = \mathfrak{L}\left[\sum_{U_j \le K} \delta_{(X_j(t),Z_j(t),\hat{U}_j)} \,\middle|\, \mathfrak{F}_t^K\right]$$

where $\hat{U}_j$ *are iid uniform on* $[0,K]$ *and independent of*

$$\sigma\big\{(X_j(r), Z_j(r), U_j) : j \in \mathbb{N}, r \le t\big\}$$

Remark 3.4.1. The interpretation of this lemma is that, for particles below level K, the past history of the empirical location/type process as embodied in the filtration $\mathfrak{F}_t^K$ provides no information about the current level values of the empirical location/type/level process.

This result is a subtle one. The filtration $\mathfrak{F}_t^K$ clearly reveals some information about the levels of particles below level K. For example, in the neutral case, we may observe two particles with dissimilar types to interact at location x at time $r \le t$. One particle changes its type to the other's, and we may then conclude that, immediately prior to time r, the particle whose type now prevails must have had the lower of the two levels. However, the two products of this interaction are indistinguishable with respect to location and type. Thus, we cannot *now* determine, as of time r, which of the two interaction products has this lower level. No interaction prior to time t can provide information about the levels *at* time t.

Proof of Lemma 3.4.3. We use the notation of Corollary 3.3.7. Define

$$\begin{aligned}\mathfrak{G}_t := \sigma\big\{X_j(0), Z_j(0), W_j, \bar{N}_j^\sigma, Y_{jk}, \zeta_{jk}, \eta_{jk} : j \in \mathbb{N} \uplus \mathbb{N}', k \in \mathbb{Z}\big\} \\ \vee \sigma\big\{U_j, N_{ij}^\lambda : i \in \mathbb{N} \uplus \mathbb{N}', j \in \mathbb{N}', i \neq j\big\} \\ \vee \sigma\Big\{N_{\{i,j\}}^\lambda, 1_{\left\{\tilde{U}_i\left(T_{\{i,j\},l}^\lambda\right) < \tilde{U}_j\left(T_{\{i,j\},l}^\lambda\right)\right\}} : i \neq j \in \mathbb{N}, l \in \{l' : T_{\{i,j\},l'}^\lambda < t\}\Big\}\end{aligned}$$

Observe that for any selective event at time τ affecting a particle at location X with level below K, there is almost surely a unique $j \in \mathbb{N}$ such that $X_j = X$ and a unique $j' \in \mathbb{N}$ such that $\tilde{X}_{j'} = X$. Thus, by Corollary 3.3.7 and Hypothesis 3.4.1, we have

$$\begin{aligned}&\psi\Big(\Big(\textstyle\sum_{i\in\mathbb{N}\uplus\mathbb{N}'} \delta_{(X_i,Z_i,U_i)}\Big), X_j(\tau), \eta\Big) \\ &\qquad = \psi\Big(\Big(\textstyle\sum_{i\in\mathbb{N}} \delta_{(\tilde{X}_i,\tilde{Z}_i,0)} + \sum_{i\in\mathbb{N}'} \delta_{(X_i,Z_i,U_i)}\Big), \tilde{X}_{j'}(\tau), \eta\Big)\end{aligned}$$

It follows that $(\tilde{X}_j, \tilde{Z}_j)_{j\in\mathbb{N}}$ is $\mathfrak{G}_t$-adapted and $\mathfrak{F}_t^K \subseteq \mathfrak{G}_t$. However, the $\{\tilde{U}_j(t)\}_{j\in\mathbb{N}}$ conditioned on $\mathfrak{G}_t$ are iid uniform on $[0, K]$. Thus, iterating the expectation, we have

$$\begin{aligned}&\mathbf{E}\left[g\left(\sum_{j\in\mathbb{N}} \delta_{(\tilde{X}_j(t),\tilde{Z}_j(t),\tilde{U}_j(t))}\right) \,\middle|\, \mathfrak{F}_t^K\right] \\ &\qquad = \mathbf{E}\left[\int g\left(\sum_{j\in\mathbb{N}} \delta_{(\tilde{X}_j(t),\tilde{Z}_j(t),u_j)}\right) H(d\mathbf{u}) \,\middle|\, \mathfrak{F}_t^K\right]\end{aligned}$$

for H the probability measure of the vector $(\hat{U}_1, \hat{U}_2, \dots)$.

By Corollary 3.3.7,

$$\sum_{j \in \mathbb{N}} \delta_{(\tilde{X}_j(t), \tilde{Z}_j(t), \tilde{U}_j(t))} =^{a.s.} \sum_{j \in \mathbb{N}} \delta_{(X_j(t), Z_j(t), U_j)}$$

and the result follows. □

These lemmas give the following theorem, which characterizes the limit of the average location/type distribution $\frac{1}{K}\langle u_t^K, h\rangle$ as $K \to \infty$.

Theorem 3.4.4. *Let* $h \in B(\mathbb{R} \times E)$ *be measurable and bounded and have compact support in its first variable. Then*

$$\tfrac{1}{K}\langle u_t^K, h\rangle = E\left[\langle u_t^1, h\rangle \mid \mathfrak{F}_t^K\right] \underset{K\to\infty}{\longrightarrow} E\left[\langle u_t^1, h\rangle \mid \mathfrak{F}_t^\infty\right]$$

where the limit is almost sure and in L_p *for all* $p \geq 1$. *We write the (integrable) limit as* $\langle u_t, h\rangle := \lim \frac{1}{K}\langle u_t^K, h\rangle$.

Proof. By Lemma 3.4.3, it follows that

$$\begin{aligned} E\left[\langle u_t^1, h\rangle \mid \mathfrak{F}_t^K\right] &= E\left[\sum_{U_j \leq K} 1_{\{U_j \leq 1\}} h(X_j(t), Z_j(t)) \,\middle|\, \mathfrak{F}_t^K\right] \\ &= E\left[\sum_{U_j \leq K} 1_{\{\hat{U}_j \leq 1\}} h(X_j(t), Z_j(t)) \,\middle|\, \mathfrak{F}_t^K\right] \\ &= \frac{1}{K}\langle u_t^K, h\rangle \end{aligned}$$

However, for $X := \langle u_t^1, h\rangle$, we have

$$0 \leq |X| \leq \|h\|_\infty \sum_{U_j \leq 1} 1_{\{|X_j(t)| \leq \alpha\}}$$

where α is such that $h(y, \cdot) \equiv 0$ for $|y| > \alpha$. By Theorem 3.5.1 of the next section, we have this summation a Poisson random variable with mean 2α. It follows that X has moments of all orders, and the result follows by Lemma 3.4.2. □

Corollary 3.4.5. *The result of Theorem 3.4.4 holds for random functions*

$$h\colon \mathbb{R} \times E \times \Omega \to \mathbb{R}$$

that are bounded and measurable with respect to $\mathfrak{B}(\mathbb{R}) \times \mathfrak{B}(E) \times \mathfrak{F}_t^\infty$ *and that have compact support in their first variable. We will use the same notation* $\langle u_t, h\rangle$ *to designate the limit for such functions.*

Proof. For $A \in \mathfrak{F}_t^\infty$ and $h \in B(\mathbb{R} \times E)$, we have

$$\mathrm{E}\left[\left\langle u_t^1, h1_A(\omega)\right\rangle \mid \mathfrak{F}_t^K\right] = 1_A(\omega)\,\mathrm{E}\left[\left\langle u_t^1, h\right\rangle \mid \mathfrak{F}_t^K\right] = \tfrac{1}{K}\left\langle u_t^K, h1_A\right\rangle$$

by the previous lemma. The result follows by approximating general h with simple functions. □

For each $t \geq 0$, let us define the random measure ν_t by

$$\nu_t(C) := \langle u_t, 1_C\rangle \tag{3.13}$$

for all $C \in \mathfrak{B}(\mathbb{R} \times E)$. Note that, at $t = 0$, this definition is consistent with the definition of the initial location/type mean measure ν_0. In Section 3.5, we will establish that the marginal location measure $\nu_t(\cdot \times E)$ is Lebesgue, and in Section 3.7, we will prove that $\{\nu_t : t \geq 0\}$ almost surely has vaguely continuous paths and thus defines a measure-valued *process*. Proposition 3.3.6 and Theorem 3.4.4 give the following result.

Proposition 3.4.6. *For each* $K > 0$*, let* $\tilde{\xi}^K$ *be given by* (1.23)*. Then, in the neutral case, we have*

$$\frac{1}{K}\tilde{\xi}_t^K \Rightarrow_{K\to\infty} \nu_t$$

Proof. By Proposition 3.3.6, we have $\tilde{\xi}_t^K =^d \xi_t^K$. But, in the neutral case

$$\xi_t^K = \xi_t(\cdot \times [0, K]) = \sum_{U_j \leq K} \delta_{(X_j(t), Z_j(t))}$$

so the result follows from Theorem 3.4.4. □

3.5 Poisson Structure

In this section, we examine the Poisson structure of the infinite-density model Υ_∞. The first theorem is a simple consequence of Lemma 1.3.2 establishing that the $Poisson(\ell_{\mathbb{R}} \times \ell_{\mathbb{R}^+})$ distribution of particles in location/level space is stationary.

Theorem 3.5.1. *For all* $t \geq 0$*, we have* $\sum_{j=1}^{\infty} \delta_{(X_j(t), u_j)}$ *a Poisson point process with mean measure* $\ell_{\mathbb{R}} \times \ell_{\mathbb{R}^+}$*.*

Proof. Note that

$$(\mathbf{X}(0), \boldsymbol{W}, \mathbf{u}) \sim Poisson(\mathbb{R} \times \mathfrak{W} \times \mathbb{R}^+)$$

for $\mathfrak{W}$ the law of W_1. The result then follows from Lemma 1.3.2 as in the proof of Corollary 1.3.3. □

In Sections 3.7 and 3.8, we will require L_1 and L_2 bounds on certain counts of particles and pairs of particles. These bounds are given by the following lemma.

Lemma 3.5.2. *Let* $\alpha > 0$ *and* $K > 0$ *be given, and define the stopping times*

$$\tau_j^\alpha := \inf\{t \geq 0 : X_j(t) \in [-\alpha, \alpha]\}$$

Then

$$\sum_{u_j \leq K} 1_{\{\tau_j^\alpha < T\}} \sim Poisson\left(K\left(2\alpha + \sqrt{\frac{2\theta T}{\pi}}\right)\right) \tag{3.14}$$

In particular, defining the constants

$$c_1 := 2\alpha + \sqrt{2\theta T/\pi}$$
$$c_2 := c_1(c_1 + K^{-1})$$

we have

$$\mathrm{E}\left(\frac{1}{K}\sum_{u_j\le K} 1_{\{\tau_j^\alpha<T\}}\right) = c_1 \tag{3.15}$$

$$\mathrm{E}\left(\frac{1}{K}\sum_{u_j\le K} 1_{\{\tau_j^\alpha<T\}}\right)^2 = c_2 \tag{3.16}$$

$$\mathrm{E}\left(\frac{1}{K^2}\sum_{u_i<u_j\le K} 1_{\{\tau_i^\alpha<T\}}1_{\{\tau_j^\alpha<T\}}\right) \le \frac{c_2}{2} \tag{3.17}$$

Proof. As observed in the proof of the previous theorem,

$$(\mathbf{X}(0), \mathbf{W}, \mathbf{U}) \sim \mathit{Poisson}(\mathbb{R}\times\mathfrak{W}\times\mathbb{R}^+)$$

Applying Lemma 1.3.2 to the map

$$h\colon (x, w, u) \mapsto \left(\inf\left\{t : \left|x+\sqrt{\theta}w_t\right| \le \alpha\right\}, u\right)$$

we see that that

$$\sum_{j=1}^{\infty}\delta_{(\tau_j^\alpha, u_j)} \sim \mathit{Poisson}(\mu\times\ell_{\mathbb{R}^+})$$

for a measure μ such that, by the reflection principle, we have

$$\begin{aligned}\mu[0,T] &= (\ell_{\mathbb{R}}\times\mathfrak{W})\left(\left\{(x_0,\mathfrak{W}) : x_0+\sqrt{\theta}w_t \text{ hits } [-\alpha,\alpha] \text{ before } T\right\}\right)\\ &= (\ell_{\mathbb{R}}\times\mathfrak{W})\left(\left\{(x_0,w) : |w_T| > (|x_0|-\alpha)/\sqrt{\theta} \text{ or } |x_0|\le\alpha\right\}\right)\\ &= 2\alpha + 2\int_\alpha^\infty P\big(Z > (x_0-\alpha)/\sqrt{\theta T}\big)\,dx_0\\ &= 2\alpha+\sqrt{2\theta T/\pi}\end{aligned}$$

This implies (3.14) and so (3.15) and (3.16). Finally, (3.17) follows from the fact that

$$\frac{2}{K^2}\sum_{u_i<u_j\le K} 1_{\{\tau_i^\alpha<T\}}1_{\{\tau_j^\alpha<T\}} \le \frac{1}{K^2}\sum_{u_i,u_j\le K} 1_{\{\tau_i^\alpha<T\}}1_{\{\tau_j^\alpha<T\}} = \left(\frac{1}{K}\sum_{u_j\le K} 1_{\{\tau_j^\alpha<T\}}\right)^2$$

□

The previous results dealt only with the Poisson structure arising from the Brownian location and fixed level processes. Of far more interest is the following result which establishes the conditional Poisson structure of the empirical location/type/level process ξ.

Theorem 3.5.3. *Let ν_t be the random measure given by* (3.13).

Then, conditioned on $\mathfrak{F}_t^\infty$, the process ξ_t is a Poisson point process with mean measure $\nu_t \times \ell_{\mathbb{R}^+}$, and ν_t almost surely has marginal location measure $\nu_t(\cdot \times E) = \ell_{\mathbb{R}}$.

Proof. Let $f \colon \mathbb{R} \times E \times \mathbb{R}^+ \to \mathbb{R}$ be bounded and measurable with compact support. Let $M > 0$ be such that $f = 0$ on $\mathbb{R} \times E \times [M, \infty)$.

For all $K \geq M$, by Lemma 3.4.3, we have

$$\begin{aligned} \mathrm{E}\left[e^{\tau \sum_j f(X_j(t), Z_j(t), U_j)} \,\middle|\, \mathfrak{F}_t^K\right] &= \mathrm{E}\left[e^{\tau \sum_{U_j \leq K} f(X_j(t), Z_j(t), U_j)} \,\middle|\, \mathfrak{F}_t^K\right] \\ &= \mathrm{E}\left[e^{\tau \sum_{U_j \leq K} f(X_j(t), Z_j(t), \hat{U}_j)} \,\middle|\, \mathfrak{F}_t^K\right] \end{aligned}$$

where $\hat{U}_j$ are iid uniform on $[0, K]$ and independent of

$$\sigma\{(X_j(r), Z_j(r), U_j) : j \in \mathbb{N}, r \leq t\}$$

It follows that

$$\begin{aligned} &\mathrm{E}\left[e^{\tau \sum_j f(X_j(t), Z_j(t), U_j)} \,\middle|\, \mathfrak{F}_t^K\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[\prod_{U_j \leq K} e^{\tau f(X_j(t), Z_j(t), \hat{U}_j)} \,\middle|\, \sigma\{(X_j(r), Z_j(r), U_j) : j \in \mathbb{N}, r \leq t\}\right] \,\middle|\, \mathfrak{F}_t^K\right] \\ &= \prod_{U_j \leq K} \frac{1}{K} \int_0^K e^{\tau f(X_j(t), Z_j(t), u)} du \end{aligned}$$

But, we may rewrite this final term and apply Taylor's formula to show that

$$\begin{aligned}
&\mathrm{E}\left[e^{\tau \sum_j f(X_j(t),Z_j(t),U_j)} \mid \mathfrak{F}_t^K\right] \\
&= \exp\left(\sum_{U_j \le K} \log\left(1 + \frac{1}{K}\int_0^\infty \left(e^{\tau f(X_j(t),Z_j(t),u)} - 1\right) du\right)\right) \\
&= \exp\left(\frac{1}{K}\left\langle u_t^K, \int_0^\infty \left(e^{\tau f(\cdot,\cdot,u)} - 1\right) du\right\rangle + O\left(\frac{1}{K}\right)\right) \\
&\underset{K\to\infty}{\longrightarrow} \exp\left(\left\langle u_t, \int_0^\infty \left(e^{\tau f(\cdot,\cdot,u)} - 1\right) du\right\rangle\right)
\end{aligned}$$

by Theorem 3.4.4.

It follows from Lemma 3.4.2 that

$$\mathrm{E}\left[e^{\tau \sum_j f(X_j(t),Z_j(t),U_j)} \mid \mathfrak{F}_t^\infty\right] = e^{\left\langle u_t, \int \left(e^{\tau f(\cdot,\cdot,u)} - 1\right) du\right\rangle}$$

which gives the conditional Poisson distribution.

Lastly, note that for all $A \in \mathfrak{B}(\mathbb{R})$, we have by Lemma 3.4.2, Theorem 3.5.1, and the strong law of large numbers that

$$\nu_t(A \times E) = \langle u_t, 1_{A\times E}\rangle = \lim_{K\to\infty} \frac{1}{K}\sum_j 1_{\{(X_j(t),U_j)\in A\times[0,K]\}} = \ell(A)$$

□

Remark 3.5.1. As the marginal location measure of ν_t is almost surely Lebesgue, it follows from Morando's theorem (Theorem A.8.1 of [5] generalized to σ-finite measures) that there exists a

$$\begin{aligned}
\hat{\nu}\colon &\mathbb{R}^+ \times \mathbb{R} \times \mathfrak{B}(E) \times \Omega \to \mathbb{R}^+ \\
&: (t, x, dz, \omega) \mapsto \hat{\nu}_t(x, dz)(\omega)
\end{aligned}$$

satisfying

$$\nu_t(C) = \int_{\mathbb{R}}\int_E 1_C(x,z)\hat{\nu}_t(x,dz)dx \tag{3.18}$$

for all $C \in \mathfrak{B}(\mathbb{R} \times E)$.

Intuitively, we may interpret $\hat{\nu}_t(x,\cdot)$ as the (random) type distribution "at" a point x in the infinite-density model.

3.6 Martingale Characterization

In this section, we will develop a martingale characterization of u_t. Specifically, we will show that for a large class of test functions h, the term $\langle u_t, h\rangle$ may be centered to form a martingale, and we will calculate the martingale's quadratic variation.

With respect to mutation, note that the Markov processes Y_{jk} are such that

$$M_{jk}^{g,y}(t) := g(Y_{jk}(y,t)) - g(y) - \int_0^t B^{\mu} g(Y_{jk}(y,s))ds$$

are martingales for all $g \in \mathfrak{D}(B^{\mu})$, $y \in E$, $j \in \mathbb{N}$, and $k \in \mathbb{Z}^+$.

With respect to selection, let us define the random measure

$$\mathfrak{V}_j^{\sigma} := \sum_k (1 - \iota_{jk})\delta_{(\tau_{jk},\zeta_{jk},\eta_{jk})}$$

Observe that each point (s, ζ, η) of $\mathfrak{V}_j^{\sigma}$ represents a potential selective event at time s affecting particle j. The η component is the random input to the function ψ used to select a type for the event, and the ζ component is used to determine whether or not particle j will actually adopt the other type. Note that we have

$$\mathfrak{V}_j^{\sigma} \sim \mathit{Poisson}(\bar{\sigma}\ell_{\mathbb{R}^+\times[0,1]\times[0,1]})$$

so that

$$\hat{\mathfrak{V}}_j^{\sigma}([0,t]\times\cdot) := \mathfrak{V}_j^{\sigma}([0,t]\times\cdot) - \bar{\sigma}t\ell(\cdot)$$

is a martingale.

For all $K > 0$, $f \in C_c^2(\mathbb{R})$ (the twice continuously differentiable functions with compact support), and $g \in \mathfrak{D}(B^{\mu})$, by Itô's formula, we have the following identity

$$
\begin{aligned}
\left\langle u_t^K, fg\right\rangle = \left\langle u_0^K, fg\right\rangle &+ \int_0^t \left\langle u_s^K, \tfrac{\theta}{2} f'' g + f B^\mu g\right\rangle ds \\
&+ \sum_{U_j \le K} \int_0^t f(X_j(s)) \int_0^1 \sigma\big(\psi(\xi_{s-}, X_j(s), \eta), Z_j(s-)\big) \\
&\qquad\qquad \Big(g\big(\psi(\xi_{s-}, X_j(s), \eta)\big) - g\big(Z_j(s-)\big)\Big)\, d\eta\, ds \\
&+ \sqrt{\theta} \sum_{U_j \le K} \int_0^t f'(X_j(s)) g(Z_j(s-))\, dW_j(s) + \sum_{U_j \le K} \int_0^t f(X_j(s))\, dM_j^g(s) \\
&+ \sum_{U_i < U_j \le K} \int_0^t \left(f(X_i(s)) g(Z_i(s-)) - f(X_j(s)) g(Z_j(s-))\right) dV_{ij}^\lambda(s) \\
&+ \sum_{U_j \le K} \int_{[0,t]\times[0,1]^2} 1_{\{\zeta \le \bar{\sigma}^{-1} \sigma(\psi(\xi_{s-}, X_j(s), \eta), Z_j(s-))\}} f(X_j(s)) \\
&\qquad\qquad \Big(g\big(\psi(\xi_{s-}, X_j(s), \eta)\big) - g\big(Z_j(s-)\big)\Big) \\
&\qquad\qquad\qquad \hat{\mathfrak{V}}_j^\sigma(ds \times d\zeta \times d\eta)
\end{aligned}
\tag{3.19}
$$

where the M_j^g are given by

$$
M_j^g(t) := \sum_k \int_0^t 1_{\{V_j(s-)=k\}} dM_{jk}^{g, Z_j(\tau_{jk}-)}(s - \tau_{jk})
$$

The independence of the $M_{jk}^{g,y}$ from the V_{jk}^λ and $\mathfrak{V}_j^\sigma$ imply that the M_j^g are $\{\mathfrak{F}_t^K\}$-martingales for all j with $U_j \le K$.

Note that the term

$$
\sum_{U_i < U_j \le K} \int_0^t \left(f(X_i(s)) g(Z_i(s-)) - f(X_j(s)) g(Z_j(s-))\right) dV_{ij}^\lambda(s) \tag{3.20}
$$

is measurable with respect to $\{\mathfrak{F}_t^K\}$. We claim it is in fact an $\{\mathfrak{F}_t^K\}$-martingale.

Lemma 3.6.1. *The term* (3.20) *is an* $\{\mathfrak{F}_t^K\}$*-martingale.*

Proof. For the proof, we use the notation of Corollary 3.3.7. Observe that

$$\begin{aligned}
&\mathrm{E}\left[\sum_{i\neq j\in\mathbb{N}} 1_{\{u_i<u_j\}}\int_s^t \left(f(X_i(r))g(Z_i(r-)) - f(X_j(r)), g(Z_j(r-))\right) dV_{ij}^{\lambda}(r) \,\middle|\, \mathfrak{F}_s^1\right] \\
&= \mathrm{E}\Bigg[\sum_{i\neq j\in\mathbb{N}}\sum_{i'\neq j'\in\mathbb{N}}\int_s^t 1_{\{\Phi_i(r-)=i'\}}1_{\{\Phi_j(r-)=j'\}}1_{\{\tilde{u}_{i'}(r-)<\tilde{u}_{j'}(r-)\}} \\
&\qquad\qquad \Big(f(\tilde{X}_{i'}(r))g(\tilde{Z}_{i'}(r-)) - f(\tilde{X}_{j'}(r))g(\tilde{Z}_{j'}(r-))\Big)\, dV_{\{i',j'\}}^{\lambda}(r) \,\Bigg|\, \mathfrak{F}_s^1\Bigg] \\
&= \mathrm{E}\left[\sum_{i'\neq j'\in\mathbb{N}}\int_s^t \Big(f(\tilde{X}_{i'}(r))g(\tilde{Z}_{i'}(r-)) - f(\tilde{X}_{j'}(r))g(\tilde{Z}_{j'}(r-))\Big)\, d\tilde{V}_{i',j'}^{\lambda}(r) \,\middle|\, \mathfrak{F}_s^1\right]
\end{aligned}$$

where the last equality follows from the definition of $\tilde{V}_{ij}^{\lambda}$ and the fact that

$$\sum_{i\neq j\in\mathbb{N}} 1_{\{\Phi_i(s-)=i'\}}1_{\{\Phi_j(s-)=j'\}} = 1$$

almost surely for all $i' \neq j' \in \mathbb{N}$. However, by symmetry in i' and j', this final term is seen to be zero. ☐

Now, for $K = 1$, the identity (3.19) implies that

$$\left\langle u_t^1, fg\right\rangle - \left\langle u_0^1, fg\right\rangle - \int_0^t \left\langle u_s^1, \tfrac{\theta}{2}f''g + fB^{\mu}g + h_s^{\sigma}\right\rangle ds$$

is an $\{\mathfrak{F}_t^1\}$-martingale for $h_s^{\sigma}\colon \mathbb{R}\times E\times\Omega \to \mathbb{R}$ given by

$$h_s^{\sigma}(x,z,\omega) := f(x)\int_0^1 \sigma\big(\psi(\xi_{s-}(\omega),x,\eta),z\big)\Big(g\big(\psi(\xi_{s-}(\omega),x,\eta)\big) - g\big(z\big)\Big)\, d\eta$$

It follows that

$$\begin{aligned}
&\mathrm{E}\left[\left\langle u_t^1, fg\right\rangle \mid \mathfrak{F}_t^{\infty}\right] - \mathrm{E}\left[\left\langle u_0^1, fg\right\rangle \mid \mathfrak{F}_0^{\infty}\right] \\
&\qquad - \int_0^t \mathrm{E}\left[\left\langle u_s^1, \tfrac{\theta}{2}f''g + fB^{\mu}g + h_s^{\sigma}\right\rangle \mid \mathfrak{F}_s^{\infty}\right] ds
\end{aligned} \tag{3.21}$$

is an $\{\mathfrak{F}_t^{\infty}\}$-martingale.

Note that the function h_s^σ depends on ω only through $\psi(\xi_{s-}(\omega), x, \eta)$. But for all $K > 0$, by Hypothesis 3.4.1,

$$\psi(\xi_{s-}(\omega), x, \eta) = \psi\left(\left(\sum_{u_j \le K} \delta_{(X_j(s), Z_j(s-), 0)} + \sum_{u_j > K} \delta_{(X_j(s), Z_j(s-), u_j)}\right), x, \eta\right)$$

which is $\mathfrak{F}_{s-}^K$-measurable. Therefore, h_s^σ satisfies the conditions of Corollary 3.4.5.

Thus, by means of Theorem 3.4.4 and its corollary, we may rewrite equation (3.21) to conclude that

$$M_t := \langle u_t, fg\rangle - \langle u_0, fg\rangle - \int_0^t \left\langle u_s, \tfrac{\theta}{2} f'' g + f B^\mu g + h_s^\sigma \right\rangle ds \tag{3.22}$$

is an $\{\mathfrak{F}_t^\infty\}$-martingale.

We now wish to determine the quadratic variation $[M]$ of this martingale. First, for each fixed $t \ge 0$, we will establish the limiting value of the angle-brackets processes $\left\langle M^K \right\rangle_t$ where M^K are the martingales given by

$$M_t^K := \frac{1}{K}\left(\left\langle u_t^K, fg\right\rangle - \left\langle u_0^K, fg\right\rangle - \int_0^t \left\langle u_s^K, \tfrac{\theta}{2} f'' g + f B^\mu g + h_s^\sigma \right\rangle ds\right) \tag{3.23}$$

Theorem 3.6.2. *For each* $t \ge 0$*, we have*

$$\left\langle M^K \right\rangle_t \xrightarrow[K\to\infty]{L^1} A_t$$

where

$$A_t := \theta\lambda \int_0^t ds \int_{\mathbb{R}} dx \int_{E\times E} \hat{\nu}_s(x, dz)\hat{\nu}_s(x, dz')\left(f(x)g(z) - f(x)g(z')\right)^2$$

for $\hat{\nu}$ *as defined in* (3.18).

Proof. We defer the proof until Section 3.8. □

We will now show that the angle-brackets process $\langle M\rangle$ is the process given by A. We begin with the following lemma.

Lemma 3.6.3. *For* $h\colon \mathbb{R} \times E \times \Omega \to \mathbb{R}$ *bounded,* $\mathfrak{B}(\mathbb{R}) \times \mathfrak{B}(E) \times \mathfrak{F}_t^\infty$*-measurable and having compact support in its first variable, we have*

$$\int_0^t \langle u_s, h\rangle \, ds = \lim_{K\to\infty} \int_0^t \frac{1}{K} \left\langle u_s^K, h\right\rangle ds$$

almost surely and in L_p *for all* $p \geq 1$*.*

Proof. Let $\alpha > 0$ be such that $h \equiv 0$ outside $[-\alpha, \alpha] \times E$. Then, for $s \in [0, t]$ and all K, we have

$$\left|\frac{1}{K} \left\langle u_s^K, h\right\rangle\right| \leq \|h\|_\infty \frac{1}{K} \sum_{U_j \leq K} 1_{\{\tau_j^\alpha < t\}}$$

with τ_j^α as defined in Lemma 3.5.2. By that lemma, we have

$$\sum_{U_j \leq K} 1_{\{\tau_j^\alpha < t\}} \sim \mathit{Poisson}(Kc_1)$$

Thus, we have

$$\frac{1}{K} \sum_{U_j \leq K} 1_{\{\tau_j^\alpha < t\}} \to c_1 \tag{3.24}$$

almost surely (and in L_p for all $p \geq 1$).

Therefore, for almost all $\omega \in \Omega$, we have $\frac{1}{K}\left\langle u_s^K, h\right\rangle$ bounded on $(s, K) \in [0, t] \times \mathbb{N}$, and for all such ω, we have

$$\int_0^t \frac{1}{K} \left\langle u_s^K, h\right\rangle ds \to \int_0^t \lim_{K\to\infty} \frac{1}{K} \left\langle u_s^K, h\right\rangle ds \tag{3.25}$$

by bounded convergence with respect to the integral $\int_{[0,t]}$.

Because the convergence in (3.24) is in L_p for all $p \geq 1$, it follows that the convergence in (3.25) is in L_p for all $p \geq 1$. □

Using this result, we can prove the following.

Lemma 3.6.4. *For all* $t \geq 0$*, we have*

$$M_t^K \xrightarrow{L^2} M_t$$

Proof. By Theorem 3.4.4 and Lemma 3.6.3, we have

$$\begin{aligned}M_t &= \langle u_t, fg\rangle - \langle u_0, fg\rangle - \int_0^t \left\langle u_s, \tfrac{\theta}{2} f'' g + fB^\mu g + h_s^\sigma \right\rangle ds \\ &= \lim_{K\to\infty} \left\{ \frac{1}{K} \left(\left\langle u_t^K, fg\right\rangle - \left\langle u_0^K, fg\right\rangle - \int_0^t \left\langle u_s^K, \tfrac{\theta}{2} f'' g + fB^\mu g + h_s^\sigma \right\rangle ds \right) \right\} \\ &= \lim_{K\to\infty} M_t^K\end{aligned}$$

in L_p for all $p \geq 1$. □

Theorem 3.6.5. *The martingale M_t has angle-brackets process*

$$\langle M\rangle = A$$

Proof. Fix $0 \leq s < t$. By Theorem 3.6.2 and Lemma 3.6.4, we have

$$0 \equiv \mathbf{E}\left[\left((M_u^K)^2 - \left\langle M^K \right\rangle_u \right) \Big|_{u=s}^{u=t} \;\Big|\; \mathfrak{F}_s^K \right] \xrightarrow[K\to\infty]{L^1} \mathbf{E}\left[\left((M_u)^2 - A_u \right) \Big|_{u=s}^{u=t} \;\Big|\; \mathfrak{F}_s^\infty \right]$$

so that $(M)^2 - A$ is a martingale. Noting that A is $\mathfrak{F}_t^\infty$-predictable, the result follows. □

In the following section, we will show that the paths of M_t are continuous and so $[M] \equiv \langle M\rangle$. Thus, the results of this section may be summarized in the following proposition.

Proposition 3.6.6. *For all $f \in C_c^2(\mathbb{R})$ and $g \in \mathfrak{D}(B^\mu)$, we have*

$$M_t := \langle u_t, fg\rangle - \langle u_0, fg\rangle - \int_0^t \left\langle u_s, \tfrac{\theta}{2} f'' g + fB^\mu g + h_s^\sigma \right\rangle ds$$

a continuous martingale with quadratic variation

$$[M]_t = \theta\lambda \int_0^t ds \int_{\mathbb{R}} dx \int_{E\times E} \hat{\nu}_s(x, dz)\hat{\nu}_s(x, dz') \left(f(x)g(z) - f(x)g(z') \right)^2$$

3.7 Continuity of the Limit

In this section, we will prove that for any $f \in C_c^2(\mathbb{R})$ with compact support $\operatorname{supp}(f) \subseteq [-\alpha, \alpha]$ and any $g \in \mathfrak{D}(B^\mu)$, the processes $\left(\frac{1}{K}\langle u_\cdot^K, fg\rangle\right)_K$ are relatively compact. The almost sure continuity of the paths of $\langle u_\cdot, fg\rangle$ then follows from Theorem 3.10.2 of [5].

Define the filtration

$$\mathfrak{F}_t := \sigma\{X_j(r), Z_j(r), U_j : j \in \mathbb{N},\ r \leq t\}$$

and let

$$\tau^\alpha := \inf\{t \geq 0 : X_j(t) \in [-\alpha, \alpha]\}$$

By Remark 3.8.7 of [5], to prove relative compactness, it is sufficient to establish that condition (a) of Theorem 3.7.2 holds, namely that for all $\eta > 0$ and $t \in \mathbb{Q}$, there exists a compact $\Gamma_{\eta,t} \subseteq \mathbb{R}$ such that

$$\inf_K P\left(\tfrac{1}{K}\langle u_t^K, fg\rangle \in \Gamma_{\eta,t}^\eta\right) \geq 1 - \eta$$

and that there exists a $\gamma^K(\delta, T)$ such for each $\delta > 0$ and $T > 0$ we have

$$E\left[\frac{1}{K^2}\left(\langle u_{t+h}^K, fg\rangle - \langle u_t^K, fg\rangle\right)^2 \,\middle|\, \mathfrak{F}_t\right] \leq E\left[\gamma^K(\delta, T) \mid \mathfrak{F}_t\right]$$

for all $0 \leq t \leq T - \delta$ and $0 \leq h \leq \delta$ and

$$\lim_{\delta\to 0} \varlimsup_{K\to\infty} E\,\gamma^K(\delta, T) = 0$$

With regards to the first condition, note that for all $\beta > 0$, we have

$$P\left(\left|\tfrac{1}{K}\langle u_t^K, fg\rangle\right| \leq \beta\right) \geq P\left(\frac{1}{K}\|f\|_\infty \|g\|_\infty \sum_{U_j \leq K} 1_{\{\tau_j^\alpha < t\}} \leq \beta\right)$$

where the summation is Poisson distributed by Lemma 3.5.2. Thus, this condition is easily satisfied by choosing a sufficiently large compact $[-\beta, \beta]$.

In the remainder of this section, we will establish that the second condition holds.

Let us write

$$h_{ij}(s) := f(X_i(s))g(Z_i(s)) - f(X_j(s))g(Z_j(s))$$

Fix $T > 0$ and $\delta > 0$. For all $0 \le t \le T - \delta$ and $0 \le h \le \delta$, by means of the identity (3.19), we have

$$\begin{aligned}
\frac{1}{25}\left(\langle u^K_{t+h}, fg\rangle - \langle u^K_t, fg\rangle\right)^2 \le \left(\int_t^{t+h} \langle u^K_s, \tfrac{\theta}{2}f''g + fB^\mu g + h^\sigma_s\rangle\, ds\right)^2 \\
+ \left(\sqrt{\theta}\sum_{u_j \le K}\int_t^{t+h} f'(X_j(s))g(Z_j(s-))dW_j(s)\right)^2 \\
+ \Bigg(\sum_{u_j \le K}\int_{[t,t+h]\times[0,1]^2} 1_{\{\zeta \le \bar{\sigma}^{-1}\sigma(\psi(\xi_{s-},X_j(s),\eta),Z_j(s-))\}} f(X_j(s)) \\
\Big(g\big(\psi(\xi_{s-}, X_j(s), \eta)\big) - g\big(Z_j(s-)\big)\Big) \\
\mathfrak{V}^\sigma_j(ds \times d\zeta \times d\eta)\Bigg)^2 \\
+ \left(\sum_{u_j \le K}\int_t^{t+h} f(X_j(s))dM^g_j(s)\right)^2 \\
+ \left(\sum_{u_i < u_j \le K}\int_t^{t+h} h_{ij}(s-)dV^\lambda_{ij}(s)\right)^2 \quad (3.26)
\end{aligned}$$

Note that

$$\begin{aligned}
E\left[\left(\int_t^{t+h} \langle u^K_s, \tfrac{\theta}{2}f''g + fB^\mu g + h^\sigma_s\rangle\, ds\right)^2 \,\middle|\, \mathfrak{F}_t\right] \\
\le E\Bigg\{\delta^2\Big(\tfrac{\theta}{2}\|f''\|_\infty\|g\|_\infty + \|f\|_\infty\|B^\mu g\|_\infty \\
+ 2\bar{\sigma}\|f\|_\infty\|g\|^2_\infty\Big)^2\Big(\sum_{u_j \le K} 1_{\{\tau^\alpha_j < T\}}\Big)^2 \,\Big|\, \mathfrak{F}_t\Bigg\} \quad (3.27)
\end{aligned}$$

and we also have

$$\mathrm{E}\left[\left(\sqrt{\theta}\sum_{U_j\le K}\int_t^{t+h} f'(X_j(s))g(Z_j(s))dW_j(s)\right)^2 \,\middle|\, \mathfrak{F}_t\right]$$

$$= \mathrm{E}\left[\theta\sum_{U_j\le K}\int_t^{t+h} (f'(X_j(s))g(Z_j(s)))^2\, ds \,\middle|\, \mathfrak{F}_t\right] \qquad (3.28)$$

$$\le \mathrm{E}\left[\theta\delta\|f'\|_\infty^2\|g\|_\infty^2 \sum_{U_j\le K} 1_{\{\tau_j^\alpha<T\}} \,\middle|\, \mathfrak{F}_t\right]$$

Similarly,

$$\mathrm{E}\left[\left(\sum_{U_j\le K}\int_{[t,t+h]\times[0,1]^2} 1_{\{\zeta\le\bar{\sigma}^{-1}\sigma(\psi(\xi_{s-},X_j(s),\eta),Z_j(s-))\}} f(X_j(s)) \right.\right.$$

$$\left.\left.\Big(g\big(\psi(\xi_{s-},X_j(s),\eta)\big)-g\big(Z_j(s-)\big)\Big)\hat{\mathfrak{V}}_j^\sigma(ds\times d\zeta\times d\eta)\right)^2 \,\middle|\, \mathfrak{F}_t\right]$$

$$= \mathrm{E}\left[\sum_{U_j\le K}\int_t^{t+h} f^2(X_j(s))\int_0^1 \sigma\left(\psi(\xi_{s-},X_j(s),\eta),Z_j(s-)\right)\right.$$

$$\left.\Big(g\big(\psi(\xi_{s-},X_j(s),\eta)\big)-g\big(Z_j(s-)\big)\Big)^2 d\eta\, ds \,\middle|\, \mathfrak{F}_t\right]$$

$$\le \mathrm{E}\left[4\delta\bar{\sigma}\|f\|_\infty^2\|g\|_\infty^2 \sum_{U_j\le K} 1_{\{\tau_j^\alpha<T\}} \,\middle|\, \mathfrak{F}_t\right]$$

To control the third term on the right-hand side of (3.26), we will need the following lemma. Observe that its condition holds under Hypothesis 1.3.1.

Lemma 3.7.1. *If for all* $g\in\mathfrak{D}(B^\mu)$ *we have* $g^2\in\mathfrak{D}(B^\mu)$*, then*

$$[M_j^g]_t-\int_0^t \left(B^\mu g^2-2gB^\mu g\right)(Z_j(s))ds$$

is a martingale.

Proof. Observe that

$$
\begin{aligned}
M_j^g(t) = g(Z_j(t)) - \int_0^t B^\mu g(Z_j(s))ds \\
- \sum_{u_i<u_j} \int_0^t \big(g(Z_i(s-)) - g(Z_j(s-))\big) dV_{ij}^\lambda(s) \\
- \int_{[0,t]\times[0,1]^2} 1_{\{\zeta\leq\bar{\sigma}^{-1}\sigma(\psi(\xi_{s-},X_j(s),\eta),Z_j(s-))\}} \\
\Big(g\big(\psi(\xi_{s-},X_j(s),\eta)\big) - g\big(Z_j(s-)\big)\Big)\mathfrak{V}_j^\sigma(ds\times d\zeta\times d\eta)
\end{aligned}
$$

Almost surely, we never have a jump attributable to mutation process and a jump attributable to a neutral or potential selective reproduction event simultaneously. Thus, we have

$$
\begin{aligned}
[M_j^g]_t = [g(Z_j)]_t - \sum_{u_i<u_j} \int_0^t \big(g(Z_i(s-)) - g(Z_j(s-))\big)^2 dV_{ij}^\lambda(s) \\
- \int_{[0,t]\times[0,1]^2} 1_{\{\zeta\leq\bar{\sigma}^{-1}\sigma(\psi(\xi_{s-},X_j(s),\eta),Z_j(s-))\}} \\
\Big(g\big(\psi(\xi_{s-},X_j(s),\eta)\big) - g\big(Z_j(s-)\big)\Big)^2\mathfrak{V}_j^\sigma(ds\times d\zeta\times d\eta)
\end{aligned}
\tag{3.29}
$$

But, we have

$$
\begin{aligned}
[g(Z_j)]_t &= g^2(Z_j(t)) - 2\int_0^t g(Z_j(s-))dg(Z_j(s)) \\
&= g^2(Z_j(t)) - 2\int_0^t gB^\mu g(Z_j(s))ds - 2\int_0^t g(Z_j(s-))dM_j^g(s) \\
&\quad - 2\sum_{u_i<u_j}\int_0^t g(Z_j(s-))\big(g(Z_i(s-)) - g(Z_j(s-))\big)dV_{ij}^\lambda(s) \\
&\quad - 2\int_{[0,t]\times[0,1]^2} 1_{\{\zeta\leq\bar{\sigma}^{-1}\sigma(\psi(\xi_{s-},X_j(s),\eta),Z_j(s-))\}} g(Z_j(s-)) \\
&\qquad \Big(g\big(\psi(\xi_{s-},X_j(s),\eta)\big) - g\big(Z_j(s-)\big)\Big)\mathfrak{V}_j^\sigma(ds\times d\zeta\times d\eta)
\end{aligned}
\tag{3.30}
$$

and we also have

$$M_j^{g^2}(t) = g^2(Z_j(t)) - \int_0^t B^\mu g^2(Z_j(s))ds$$
$$- \sum_{u_i<u_j} \int_0^t \big(g^2(Z_i(s-)) - g^2(Z_j(s-))\big) dV_{ij}^\lambda(s)$$
$$- \int_{[0,t]\times[0,1]^2} 1_{\{\zeta\le\bar{\sigma}^{-1}\sigma(\psi(\xi_{s-},X_j(s),\eta),Z_j(s-))\}}$$
$$\Big(g^2\big(\psi(\xi_{s-},X_j(s),\eta)\big) - g^2\big(Z_j(s-)\big)\Big)\mathfrak{V}_j^\sigma(ds\times d\zeta\times d\eta) \quad (3.31)$$

Combining (3.29), (3.30), and (3.31), the reproductive event terms cancel, and we have

$$[M_j^g]_t - \int_0^t \big(B^\mu g^2 - 2gB^\mu g\big)(Z_j(s))ds = M_j^{g^2}(t) - 2\int_0^t g(Z_j(s-))dM_j^g(s)$$

which is a martingale. □

By the lemma,

$$\mathbf{E}\left[\left(\sum_{u_j\le K}\int_t^{t+h} f(X_j(s))dM_j^g(s)\right)^2 \middle| \mathfrak{F}_t\right]$$
$$= \mathbf{E}\left[\sum_{u_j\le K}\int_t^{t+h} f^2(X_j(s))d[M_j^g]_s \middle| \mathfrak{F}_t\right]$$
$$= \mathbf{E}\left[\sum_{u_j\le K}\int_t^{t+h} f^2(X_j(s))(B^\mu g^2 - 2gB^\mu g)(Z_j(s))ds \middle| \mathfrak{F}_t\right]$$
$$\le \mathbf{E}\left[\delta\|f\|_\infty^2\|B^\mu g^2 - 2gB^\mu g\|_\infty \sum_{u_j\le K} 1_{\{\tau_j^\alpha<T\}} \middle| \mathfrak{F}_t\right] \quad (3.32)$$

Let us define the stopping time

$$\tau_{ij}^\alpha := \inf\{s>0 : X_i(s) = X_j(s), |X_j(s)|\le\alpha\}$$

Noting that

$$\sum_{u_i<u_j\le K}\int_t^{t+h} h_{ij}(s-)dV_{ij}^\lambda(s)$$

is a martingale by Lemma 3.6.1, we have

$$\begin{aligned} E&\left[\left(\sum_{u_i<u_j\le K}\int_t^{t+h} h_{ij}(s)\,dV_{ij}^{\lambda}(s)\right)^2 \,\middle|\, \mathfrak{F}_t\right] \\ &= E\left[\frac{\lambda}{2\theta}\sum_{u_i<u_j\le K}\int_t^{t+h} h_{ij}^2(s)\,dL_{ij}(s) \,\middle|\, \mathfrak{F}_t\right] \\ &\le E\left[2\lambda\theta^{-1}\|f\|_\infty^2\|g\|_\infty^2 \sum_{u_i<u_j\le K} 1_{\{\tau_{ij}^\alpha<T\}} \sup_{\tau_{ij}^\alpha\le s\le T}(L_{ij}(s+\delta)-L_{ij}(s)) \,\middle|\, \mathfrak{F}_t\right] \end{aligned} \tag{3.33}$$

Defining

$$\begin{aligned} \gamma^K(\delta,T) := \frac{25}{K^2}\Bigg\{&\delta^2\left(\tfrac{\theta}{2}\|f''\|_\infty\|g\|_\infty + \|f\|_\infty\|B^\mu g\|_\infty\right)^2\left(\sum_{u_j\le K} 1_{\{\tau_j^\alpha<T\}}\right)^2 \\ &+\Big(\theta\delta\|f'\|_\infty^2\|g\|_\infty^2 + 4\delta\bar{\sigma}\|f\|_\infty^2\|g\|_\infty^2 \\ &\qquad + \delta\|f\|_\infty^2\|B^\mu g^2 - 2gB^\mu g\|_\infty\Big)\sum_{u_j\le K} 1_{\{\tau_j^\alpha<T\}} \\ &+ 2\lambda\theta^{-1}\|f\|_\infty^2\|g\|_\infty^2 \sum_{u_i<u_j\le K} 1_{\{\tau_{ij}^\alpha<T\}} \sup_{\tau_{ij}^\alpha\le s\le T}(L_{ij}(s+\delta)-L_{ij}(s))\Bigg\} \end{aligned} \tag{3.34}$$

we may combine inequalities (3.26), (3.27), (3.28), (3.32), and (3.33) to see that

$$E\left[\frac{1}{K^2}\left(\langle u_{t+h}^K, fg\rangle - \langle u_t^K, fg\rangle\right)^2 \,\middle|\, \mathfrak{F}_t\right] \le E\left[\gamma^K(\delta,T) \mid \mathfrak{F}_t\right]$$

Now, let us define

$$\lambda(\delta,t) := E\sup_{0\le s\le t}\left(\mathfrak{l}_{2\theta(s+\delta)} - \mathfrak{l}_{2\theta s}\right)$$

where $\mathfrak{l}$ is the local time at zero of a standard Brownian motion started at zero. Note that by the strong Markov property, we have

$$\begin{aligned}
&\mathrm{E}\left[1_{\{\tau_{ij}^{\alpha}<T\}} \sup_{\tau_{ij}^{\alpha}\le s\le T}(L_{ij}(s+\delta)-L_{ij}(s)) \,\middle|\, \sigma(X_j(0),U_j : j\in\mathbb{N})\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[1_{\{\tau_{ij}^{\alpha}<T\}} \sup_{\tau_{ij}^{\alpha}\le s\le T}(L_{ij}(s+\delta)-L_{ij}(s)) \,\middle|\, \mathfrak{F}_{\tau_{ij}^{\alpha}}^{X_i,X_j}\right] \,\middle|\, \sigma(X_j(0),U_j : j\in\mathbb{N})\right] \\
&= \mathrm{E}\left[1_{\{\tau_{ij}^{\alpha}<T\}}\lambda(\delta, T-\tau_{ij}^{\alpha}) \,\middle|\, \sigma(X_j(0),U_j : j\in\mathbb{N})\right] \\
&\le \lambda(\delta,T)\,\mathrm{E}\left[1_{\{\tau_i^{\alpha}<T\}}1_{\{\tau_j^{\alpha}<T\}} \,\middle|\, \sigma(X_j(0),U_j : j\in\mathbb{N})\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathrm{E}\left(\sum_{U_i<U_j\le K} 1_{\{\tau_{ij}^{\alpha}<T\}} \sup_{\tau_{ij}^{\alpha}\le s\le T}(L_{ij}(s+\delta)-L_{ij}(s))\right) \\
&\qquad= \mathrm{E}\left(\sum_{U_i<U_j\le K} \mathrm{E}\left[1_{\{\tau_{ij}^{\alpha}<T\}} \sup_{\tau_{ij}^{\alpha}\le s\le T}(L_{ij}(s+\delta)-L_{ij}(s)) \,\middle|\, \sigma(X_j(0),U_j : j\in\mathbb{N})\right]\right) \\
&\qquad\le \lambda(\delta,T)\,\mathrm{E}\left(\sum_{U_i<U_j\le K} 1_{\{\tau_i^{\alpha}<T\}}1_{\{\tau_j^{\alpha}<T\}}\right)
\end{aligned}$$

Finally, then, it follows that from Lemma 3.5.2 that for all $K > 1$, we have

$$\begin{aligned}
\mathrm{E}\,\gamma^K(\delta,T) \le 25\Bigg\{ &\delta^2\left(\tfrac{\theta}{2}\|f''\|_\infty\|g\|_\infty + \|f\|_\infty\|B^\mu g\|_\infty\right)^2 c_1(1+c_1) \\
&+ \left(\theta\delta\|f'\|_\infty^2\|g\|_\infty^2 + 4\delta\bar{\sigma}\|f\|_\infty^2\|g\|_\infty^2 + \delta\|f\|_\infty^2\|B^\mu g^2 - 2gB^\mu g\|_\infty\right)\frac{c_1}{K} \\
&+ 2\lambda\theta^{-1}\|f\|_\infty^2\|g\|_\infty^2\lambda(\delta,T)c_1(1+c_1)\Bigg\}
\end{aligned}$$

Now, by the continuity and monotonicity of $\mathfrak{l}$, it follows that

$$\sup_{0\le s\le T}\left(\mathfrak{l}_{2\theta(s+\delta)} - \mathfrak{l}_{2\theta s}\right) \xrightarrow[\delta\to 0]{a.s.} 0$$

However, this supremum is bounded by the L_1 random variable $\mathfrak{l}_{2\theta(T+\delta)}$, so by the dominated convergence theorem, we have $\lambda(\delta, T) \longrightarrow_{\delta\to 0} 0$.

From this, it follows that

$$\lim_{\delta\to 0} \overline{\lim_{K\to\infty}} \, \mathrm{E}\, \gamma^K(\delta, T) = 0$$

and the relative compactness and continuity of $\left(\frac{1}{K}\left\langle u^K, fg\right\rangle\right)_K$ are established.

Remark 3.7.1. Observe that for the martingales M^K defined in (3.23), we have (cf. inequality (3.26))

$$\begin{aligned}
&\frac{1}{16}\left(M^K_{t+h} - M^K_t\right)^2 \\
&\leq \left(\sqrt{\theta} \sum_{u_j \leq K} \int_t^{t+h} f'(X_j(s)) g(Z_j(s-)) \, dW_j(s)\right)^2 \\
&\quad + \Bigg(\sum_{u_j\leq K} \int_{[t,t+h]\times[0,1]^2} 1_{\left\{\zeta \leq \bar{\sigma}^{-1}\sigma\left(\psi(\xi_{s-}, X_j(s), \eta), Z_j(s-)\right)\right\}} f(X_j(s)) \\
&\qquad\qquad \left(g\left(\psi(\xi_{s-}, X_j(s), \eta)\right) - g\left(Z_j(s-)\right)\right) \\
&\qquad\qquad\qquad \hat{\mathfrak{V}}^{\sigma}_j(ds \times d\zeta \times d\eta)\Bigg)^2 \\
&\qquad + \left(\sum_{u_j\leq K} \int_t^{t+h} f(X_j(s)) \, dM^g_j(s)\right)^2 \\
&\qquad\qquad + \left(\sum_{u_i < u_j \leq K} \int_t^{t+h} h_{ij}(s-) \, dV^{\lambda}_{ij}(s)\right)^2
\end{aligned}$$

and so the proof above also establishes relative compactness of $(M^K)_K$ and the continuity of the limiting martingale M defined in (3.22).

3.8 Calculation of the Quadratic Variation

In this section, we will prove Theorem 3.6.2, completing the martingale characterization of Section 3.6.

Let us write

$$h_{ij}(s) := f(X_i(s))g(Z_i(s)) - f(X_j(s))g(Z_j(s))$$

Note that, by (3.19), we have

$$\begin{aligned}
M_t^K &= \frac{1}{K}\left(\langle u_t^K, fg\rangle - \langle u_0^K, fg\rangle - \int_0^t \langle u_s^K, \tfrac{\theta}{2}f''g + fB^\mu g + h_s^\sigma\rangle\, ds\right)\\
&= \frac{1}{K}\Bigg(\sqrt{\theta}\sum_{u_j\le K}\int_0^t f'(X_j(s))g(Z_j(s-))dW_j(s)\\
&\quad + \sum_{u_j\le K}\int_{[0,t]\times[0,1]^2} 1_{\{\zeta\le\bar{\sigma}^{-1}\sigma(\psi(\xi_{s-},X_j(s),\eta),Z_j(s-))\}} f(X_j(s))\\
&\qquad \Big(g\big(\psi(\xi_{s-},X_j(s),\eta)\big) - g\big(Z_j(s-)\big)\Big)\mathfrak{V}_j^\sigma(ds\times d\zeta\times d\eta)\\
&\qquad + \sum_{u_j\le K}\int_0^t f(X_j(s))dM_j^g(s) + \sum_{u_i<u_j\le K}\int_0^t h_{ij}(s-)dV_{ij}^\lambda(s)\Bigg)
\end{aligned}$$

The first three terms on the right-hand side are martingales, and the last is also a martingale by Lemma 3.6.1. Therefore, we have

$$\begin{aligned}
\langle M^K\rangle_t &= \frac{\theta}{K^2}\sum_{u_j\le K}\int_0^t \big(f'(X_j(s))g(Z_j(s-))\big)^2 ds\\
&\quad + \sum_{u_j\le K}\int_0^t f^2(X_j(s))\int_0^1 \sigma\left(\psi(\xi_{s-},X_j(s),\eta), Z_j(s-)\right)\\
&\qquad \Big(g\big(\psi(\xi_{s-},X_j(s),\eta)\big) - g\big(Z_j(s-)\big)\Big)^2 d\eta\, ds\\
&\quad + \frac{1}{K^2}\sum_{u_j\le K}\int_0^t f^2(X_j(s))\,d\left\langle M_j^g\right\rangle_s + \frac{\lambda}{2\theta K^2}\sum_{u_i<u_j\le K}\int_0^t h_{ij}^2(s-)dL_{ij}(s)
\end{aligned}$$

As $K\to\infty$, the first three terms converge in L_1 to zero by arguments similar to those used in the last section. It remains only to show that the final term converges in L_1 to the process A defined in Theorem 3.6.2.

For each $K>0$, $n\in\mathbb{N}$, define

$$
\begin{aligned}
H_K \;&:= \frac{1}{K^2} \sum_{U_i < U_j \le K} \int_0^t h_{ij}^2(s)\, dL_{ij}(s) \\
H_{K,n} &:= \frac{1}{K^2} \sum_{U_i < U_j \le K} \int_0^t h_{ij}^2(s) \int_{\mathbb{R}} n^2 1_{\{X_i(s), X_j(s) \in [x, x+\frac{1}{n})\}} 2\theta ds \\
H_n \;&:= \int_0^t \int_{\mathbb{R}} \Bigg(n \int_x^{x+\frac{1}{n}} \int_E f^2(y) g^2(z) \hat{\nu}_s(y, dz) dy \\
&\qquad\qquad - \left(n \int_x^{x+\frac{1}{n}} \int_E f(y) g(z) \hat{\nu}_s(y, dz) dy \right)^2 \Bigg) dx\, 2\theta ds \\
H \;&:= \int_0^t \int_{\mathbb{R}} \Bigg(\int_E f^2(x) g^2(z) \hat{\nu}_s(x, dz) \\
&\qquad\qquad - \left(\int_E f(x) g(z) \hat{\nu}_s(x, dz) \right)^2 \Bigg) dx\, 2\theta ds
\end{aligned}
$$

Noting that $H = 2\theta\lambda^{-1}A$, we need only prove that $H_K \xrightarrow{L^1} H$. To do this, we will establish that:

- $\gamma_n := \overline{\lim}_K \mathrm{E}\, |H_K - H_{K,n}| \longrightarrow_{n\to\infty} 0$;
- for all n, $H_{K,n} \xrightarrow{L^1}_{K\to\infty} H_n$;
- and $H_n \xrightarrow{L^1}_{n\to\infty} H$.

Then, it will follow that

$$
\begin{aligned}
\overline{\lim_K} \mathrm{E}\, |H_K - H| &\le \overline{\lim_K} \mathrm{E}\, |H_K - H_{K,n}| + \overline{\lim_K} \mathrm{E}\, |H_{K,n} - H_n| + \mathrm{E}\, |H_n - H| \\
&= \gamma_n + \mathrm{E}\, |H_n - H| \\
&\underset{n\to\infty}{\longrightarrow} 0
\end{aligned}
$$

giving the desired convergence.

Let us begin by establishing the convergence of γ_n in the following lemma.

Lemma 3.8.1.

$$
\gamma_n := \overline{\lim_K} \mathrm{E}\, |H_K - H_{K,n}| \underset{n\to\infty}{\longrightarrow} 0
$$

Proof. Let k_n be a $C^2(\mathbb{R})$ approximation of $|\cdot|$ given by

$$k_n(x) = \begin{cases} |x|, & x \notin [-\frac{1}{n}, \frac{1}{n}]; \\ |x| + \frac{n^2(\frac{1}{n}-|x|)^3}{3}, & x \in [-\frac{1}{n}, \frac{1}{n}]. \end{cases}$$

Note that $\big|k_n(x) - |x|\big| \leq 1/3n$ for all $x \in \mathbb{R}$ and that $|k_n'(x) - \mathrm{sgn}(x)| = 0$ except on $[-\frac{1}{n}, \frac{1}{n}]$ where it is bounded by 1. Also observe that

$$\begin{aligned} L^n(t) &:= \int_0^t n^2 \int_{\mathbb{R}} 1_{\{X_i(s), X_j(s) \in [x, x+\frac{1}{n})\}} dx \, 2\theta ds \\ &= \int_0^t \big(n(1 - n|X_i(s) - X_j(s)|) \vee 0\big) 2\theta ds \\ &= \frac{1}{2} \int_0^t k_n''(X_{ij}(s)) d[X_{ij}]_s \end{aligned}$$

so by the Meyer-Itô Formula [10, Theorem IV.51], we have

$$\begin{aligned} \big(k_n(X_{ij}(t)) - |X_{ij}(t)|\big) &= \big(k_n(X_{ij}(0)) - |X_{ij}(0)|\big) \\ &\quad + \int_0^t \big(k_n'(X_{ij}(s)) - \mathrm{sgn}(X_{ij}(s))\big) dX_{ij}(s) + L_{ij}^n(t) - L_{ij}(t) \end{aligned} \tag{3.35}$$

Since, by the product rule, we have

$$\begin{aligned} \big(k_n(X_{ij}(t)) - |X_{ij}(t)|\big) h_{ij}^2(t) &= \big(k_n(X_{ij}(0)) - |X_{ij}(0)|\big) h_{ij}^2(0) \\ &\quad + \int_0^t h_{ij}^2(s-) d\big(k_n(X_{ij}(s)) - |X_{ij}(s)|\big) \\ &\quad + \int_0^t \big(k_n(X_{ij}(s)) - |X_{ij}(s)|\big) d(h_{ij}^2)(s) \\ &\qquad + \big[k_n(X_{ij}) - |X_{ij}|, h_{ij}^2\big]_t \end{aligned}$$

it follows that

$$\int_0^t h_{ij}^2(s)\,d\left(L_{ij}^n - L_{ij}\right)(s) =$$
$$\left(k_n(X_{ij}(t)) - |X_{ij}(t)|\right)h_{ij}^2(t) - \left(k_n(X_{ij}(0)) - |X_{ij}(0)|\right)h_{ij}^2(0)$$
$$- \int_0^t h_{ij}^2(s-)\left(k_n'(X_{ij}(s)) - \operatorname{sgn}(X_{ij}(s))\right)dX_{ij}(s)$$
$$- \int_0^t \left(k_n(X_{ij}(s)) - |X_{ij}(s)|\right)d(h_{ij}^2)(s)$$
$$- \left[k_n(X_{ij}) - |X_{ij}|, h_{ij}^2\right]_t \quad (3.36)$$

Now, as

$$\gamma_n = \overline{\lim_K}\, \mathrm{E}\left|\frac{1}{K^2}\sum_{u_i<u_j\le K}\int_0^t h_{ij}^2(s)\,d\left(L_{ij}^n - L_{ij}\right)(s)\right|$$

we may establish the lemma by establishing the limit

$$\lim_n \overline{\lim_K}\, \mathrm{E}\left|\frac{1}{K^2}\sum_{u_i<u_j\le K}\left(\,\cdot\,\right)\right| = 0$$

for each term on the right-hand side of (3.36).

For the first term, defining $\tau_j^\alpha := \inf\{t > 0 : |X_j(t)| > \alpha\}$, we see that

$$\mathrm{E}\left|\frac{1}{K^2}\sum_{u_i<u_j\le K}\left(k_n(X_{ij}(t)) - |X_{ij}(t)|\right)h_{ij}^2(t)\right|$$
$$\le \mathrm{E}\left[\frac{1}{K^2}\sum_{u_i<u_j\le K}\frac{4\|f\|_\infty^2\|g\|_\infty^2}{n}1_{\left\{\substack{\tau_i^\alpha<t\\ \tau_j^\alpha<t}\right\}}\right]$$
$$\le \frac{2\|f\|_\infty^2\|g\|_\infty^2}{n}c_2 \underset{n\to\infty}{\longrightarrow} 0$$

by (3.17) of Lemma 3.5.2, and the same argument holds for the second term.

For the third term, note that

$$M_t := \frac{1}{K^2} \sum_{U_i < U_j \le K} \int_0^t h_{ij}^2(s-) \left(k_n'(X_{ij}(s)) - \operatorname{sgn}(X_{ij}(s))\right) dX_{ij}(s)$$

is a martingale with quadratic variation

$$\begin{aligned}
[M]_t &= \frac{1}{K^4} \sum_{U_i < U_j \le K} \int_0^t h_{ij}^4(s-) \left(k_n'(X_{ij}(s)) - \operatorname{sgn}(X_{ij}(s))\right)^2 2\theta ds \\
&\le \frac{1}{6} K^4 \sum_{U_i < U_j \le K} 1_{\left\{ \substack{\tau_i^\alpha < t \\ \tau_j^\alpha < t} \right\}} \|f\|_\infty^4 \|g\|_\infty^4 2\theta t
\end{aligned}$$

Therefore, by Jensen's inequality, we have

$$\mathrm{E}\,|M_t| \le (\mathrm{E}\, M_t^2)^{\frac{1}{2}} = (\mathrm{E}[M]_t)^{\frac{1}{2}} \le \left(\frac{1}{K^2} \|f\|_\infty^4 \|g\|_\infty^4 \theta t c_2\right)^{\frac{1}{2}} \underset{K \to \infty}{\longrightarrow} 0$$

For the fourth term, note that, by Itô's formula, we may write $h_{ij}^2(t)$ as the sum

$$h_{ij}^2(t) = h_{ij}^2(0) + A_{ij}(t) + M_{ij}(t) + \int_0^t h_{ij}(s-) d(J_i + J_j)(s) \tag{3.37}$$

for a finite variation process

$$\begin{aligned}
A_{ij}(t) := \int_0^t \Big(& h_{ij}(s-)\big(f''(X_i(s)) g(Z_i(s-)) - f''(X_j(s)) g(Z_j(s-)) \\
& + 2f(X_i(s)) B^\mu g(Z_i(s)) - 2f(X_j(s)) B^\mu g(Z_j(s))\big) \\
& + \big(f'(X_i(s)) g(Z_i(s-))\big)^2 + \big(f'(X_j(s)) g(Z_j(s-))\big)^2 \Big) ds
\end{aligned}$$

a martingale

$$\begin{aligned}
M_{ij}(t) := 2 \int_0^t h_{ij}(s-) \Big(& f'(X_i(s)) g(Z_i(s-)) dX_i(s) \\
& - f'(X_j(s)) g(Z_j(s-)) dX_j(s) \\
& + f(X_i(s)) dM_i^g(s) - f(X_j(s)) dM_j^g(s) \Big)
\end{aligned}$$

and a jump process

$$J_j(t) := \sum_k 1_{\{U_k<U_j\}}\big(g(Z_k(s-)) - g(Z_j(s-))\big)V^{\lambda}_{kj}(s) + \int_{[0,t]\times[0,1]^2} 1_{\left\{\zeta\le\bar\sigma^{-1}\sigma\left(\psi(\xi_{s-},X_j(s),\eta),Z_j(s-)\right)\right\}} \Big(g\big(\psi(\xi_{s-},X_j(s),\eta)\big) - g\big(Z_j(s-)\big)\Big)\mathfrak{V}^{\sigma}_j(ds\times d\zeta\times d\eta)$$

By Tonelli's theorem, we have

$$\mathrm{E}\left|\frac{1}{K^2}\sum_{U_i<U_j\le K}\int_0^t\big(k_n(X_{ij}(s)) - |X_{ij}(s)|\big)\,dA_{ij}(s)\right| \le \mathrm{E}\,\frac{1}{K^2}\sum_{U_i<U_j\le K}\frac{1}{n}|A_{ij}|(t)$$
$$\le \frac{t}{n}C_1\,\mathrm{E}\,\frac{1}{K^2}\sum_{U_i<U_j\le K} 1_{\left\{\substack{\tau_i^\alpha<t\\ \tau_j^\alpha<t}\right\}} \le \frac{t}{n}C_1\frac{c_2}{2}\underset{n\to\infty}{\longrightarrow} 0$$

where

$$C_1 = 4\|f\|_\infty\|g\|_\infty\big(\|f''\|_\infty\|g\|_\infty + 2\|f\|_\infty\|B^\mu g\|_\infty\big) + 2\|f'\|^2_\infty\|g\|^2_\infty < \infty$$

We also have

$$M_t := \frac{1}{K^2}\sum_{U_i<U_j\le K}\int_0^t\left(k_n(X_{ij}(s)) - |X_{ij}(s)|\right)dM_{ij}(s)$$

a martingale whose quadratic variation has expectation satisfying

$$\mathrm{E}[M]_t = \mathrm{E}\,\frac{1}{K^4}\sum_{U_i<U_j\le K}\int_0^t\left(k_n(X_{ij}(s)) - |X_{ij}(s)|\right)^2 d[M_{ij}](s)$$
$$\le \frac{16}{n^2K^4}\|f\|^2_\infty\|g\|^2_\infty\sum_{U_i<U_j\le K}\big(2\theta\|f'\|^2_\infty\|g\|^2_\infty t + 2\|f\|^2_\infty\|B^\mu g^2 - 2gB^\mu g\|_\infty\big)1_{\left\{\substack{\tau_i^\alpha<t\\ \tau_j^\alpha<t}\right\}}$$

by Lemma 3.7.1. By Jensen's inequality, we have

$$\mathrm{E}\,|M_t| \le (\mathrm{E}\,M_t^2)^{\frac{1}{2}} = (\mathrm{E}[M]_t)^{\frac{1}{2}}\underset{K\to\infty}{\longrightarrow} 0$$

The term involving the jump processes J_j may be similarly controlled, and it follows that

$$\lim_n \overline{\lim_K} \mathrm{E} \frac{1}{K^2} \sum_{U_i < U_j \leq K} \int_0^t (k_n(X_{ij}(s)) - |X_{ij}(s)|) \, d(h_{ij}^2)(s) = 0$$

Finally, for the fifth term, equation (3.35) in combination with the decomposition (3.37) implies that

$$\begin{aligned}
&\left[k_n(X_{ij}) - |X_{ij}|, h_{ij}^2\right]_t \\
&= \Bigg[\int_0^t \big(k'(X_{ij}(s)) - \mathrm{sgn}(X_{ij}(s))\big) dX_{ij}(s), \\
&\qquad 2\int_0^t h_{ij}(s-)\Big(f'(X_i(s))g(Z_i(s-))dX_i(s) - f'(X_j(s))g(Z_j(s-))dX_j(s) \\
&\qquad\qquad + f(X_i(s))dM_i^g(s) - f(X_j(s))dM_j^g(s)\Big)\Bigg]_t \\
&= 2\int_0^t \big(k'(X_{ij}(s)) - \mathrm{sgn}(X_{ij}(s))\big) h_{ij}(s-) \\
&\qquad\qquad \Big(f'(X_i(s))g(Z_i(s-)) - f'(X_j(s))g(Z_j(s-))\Big)\theta ds \\
&\leq 8\theta\|f\|_\infty\|f'\|_\infty\|g\|_\infty^2 \int_0^t 1_{\{X_{ij}(s) \leq \frac{1}{n}\}} 1_{\{|X_i(s)|,|X_j(s)| \leq \alpha\}} ds
\end{aligned}$$

so we see by Tonelli's theorem that

$$\begin{aligned}
&\mathrm{E}\left|\frac{1}{K^2} \sum_{U_i < U_j \leq K} \left[k_n(X_{ij}) - |X_{ij}|, h_{ij}^2\right]_t\right| \\
&\leq 8\theta\|f\|_\infty\|f'\|_\infty\|g\|_\infty^2 \,\mathrm{E} \int_0^t \frac{1}{K^2} \sum_{U_i < U_j \leq K} 1_{\{X_{ij}(s) \leq \frac{1}{n}\}} 1_{\{|X_i(s)|,|X_j(s)| \leq \alpha\}} ds \\
&\leq 8\theta\|f\|_\infty\|f'\|_\infty\|g\|_\infty^2 \,\mathrm{E} \int_0^t n \int_{-\alpha}^{\alpha} \frac{1}{K^2} \sum_{U_i < U_j \leq K} 1_{\{|X_i(s)|,|X_j(s)| \in [x - \frac{1}{n}, x + \frac{1}{n}]\}} dx \, ds \\
&\leq 8\theta\|f\|_\infty\|f'\|_\infty\|g\|_\infty^2 \int_0^t n \int_{-\alpha}^{\alpha} \mathrm{E} \frac{1}{K^2} \sum_{U_i < U_j \leq K} 1_{\{|X_i(s)|,|X_j(s)| \in [x - \frac{1}{n}, x + \frac{1}{n}]\}} dx \, ds
\end{aligned}$$

However, this last expectation is $1/K^2$ times the expected number of unordered pairs in location-level space $[x-\frac{1}{n}, x+\frac{1}{n}] \times [0,K]$ and so has value

$$\mathrm{E}\,\frac{1}{K^2}\sum_{U_i<U_j\leq K} 1_{\{|X_i(s)|,|X_j(s)|\in[x-\frac{1}{n},x+\frac{1}{n}]\}} \leq n^{-1}(K^{-1}+2n^{-1})$$

Finally, then

$$\mathrm{E}\left|\frac{1}{K^2}\sum_{U_i<U_j\leq K}\left[k_n(X_{ij})-|X_{ij}|, h_{ij}^2\right]_t\right| \leq 16\theta\|f\|_\infty\|f'\|_\infty\|g\|_\infty^2 t\alpha(K^{-1}+2n^{-1})$$

and we have the result. □

Lemma 3.8.2. *For each* $n \in \mathbb{N}$,

$$H_{K,n} \xrightarrow[K\to\infty]{L^1} H_n$$

Proof. By Tonelli's theorem,

$$\mathrm{E}\,|H_{K,n}-H_n| = \mathrm{E}\left|\int_0^t\int_{\mathbb{R}}\left(V_{s,x}^K-\sigma_{s,x}^2\right)dx\,2\theta ds\right| \tag{3.38}$$

where

$$V_{s,x}^K := \frac{n^2}{K^2}\sum_{U_i<U_j\leq K} 1_{\{X_i(s),X_j(s)\in[x,x+\frac{1}{n})\}}h_{ij}^2(s)$$

$$\sigma_{s,x}^2 := \left(n\int_x^{x+\frac{1}{n}}\int_E f^2(y)g^2(z)\hat{\nu}_s(y,dz)dy - \left(n\int_x^{x+\frac{1}{n}}\int_E f(y)g(z)\hat{\nu}_s(y,dz)dy\right)^2\right)$$

However, noting that

$$\sum_j 1_{\{X_j(s)\in[x,x+\frac{1}{n})\}}\delta_{(X_j(s),Z_j(s),U_j)} \sim \textit{Poisson}(\mu)$$

for μ given by

$$\mu(A\times B\times C) := \ell(C)\int_{A\cap[x,x+\frac{1}{n})}\hat{\nu}_s(x,B)dx$$

let us reorder the points

$$\sum_j \delta_{(\hat{X}_j(s),\hat{Z}_j(s),\hat{u}_j)} := \sum_j 1_{\{X_j(s)\in[x,x+\frac{1}{n})\}}\delta_{(X_j(s),Z_j(s),u_j)}$$

so that the levels $\hat{u}_j$ are strictly increasing. Then, note that for all K, we have

$$\sum_{u_j\le K} 1_{\{X_j(s)\in[x,x+\frac{1}{n})\}}\delta_{(X_j(s),Z_j(s))} = \sum_{j=1}^{N^K_{s,x}} \delta_{(\hat{X}_j(s),\hat{Z}_j(s))}$$

with $N^K_{s,x} \sim \textit{Poisson}\left(\frac{K}{n}\right)$ and the $(\hat{X}_j(s),\hat{Z}_j(s))$ iid and independent of $N^K_{s,x}$ with distribution given by

$$n1_{[x,x+\frac{1}{n})}\hat{\nu}_s(x,dz)\,dx$$

Therefore, by the strong law of large numbers, we have

$$\begin{aligned}
&V^K_{s,x}\\
&= \frac{n^2}{2K^2}\sum_{i=1}^{N^K_{s,x}}\sum_{j=1}^{N^K_{s,x}}\left(f(\hat{X}_i(s))g(\hat{Z}_i(s)) - f(\hat{X}_j(s))g(\hat{Z}_j(s))\right)^2\\
&= \frac{n^2}{2K^2}\left(N^K_{s,x}\sum_{j=1}^{N^K_{s,x}} 2f^2(\hat{X}_j(s))g^2(\hat{Z}_j(s)) - 2\left(\sum_{j=1}^{N^K_{s,x}} f(\hat{X}_j(s))g(\hat{Z}_j(s))\right)^2\right)\\
&= \left(\frac{nN^K_{s,x}}{K}\right)\left(\frac{1}{N^K_{s,x}}\sum_{j=1}^{N^K_{s,x}} f^2(\hat{X}_j(s))g^2(\hat{Z}_j(s)) - \left(\frac{1}{N^K_{s,x}}\sum_{j=1}^{N^K_{s,x}} f(\hat{X}_j(s))g(\hat{Z}_j(s))\right)^2\right)\\
&\to \mathbb{E}\, f^2(\hat{X}_1(s))g^2(\hat{Z}_1(s)) - \left(\mathbb{E}\, f(\hat{X}_1(s))g(\hat{Z}_1(s))\right)^2\\
&= \sigma^2_{s,x}
\end{aligned}$$

for all $s \in \mathbb{R}^+$, all $x \in \mathbb{R}$, and almost all $\omega \in \Omega$.

Since

$$
\begin{aligned}
&|V^K_{s,x} - \sigma^2_{s,x}| \\
&\leq 1_{\{|x|\leq \alpha+\frac{1}{n}\}}\|f\|^2_\infty \|g\|^2_\infty \left(4n^2 \sum_{i,j} \frac{1}{K^2} 1_{\{U_i<U_j\leq K\}} 1_{\{\tau^\alpha_i<t\}} 1_{\{\tau^\alpha_j<t\}} + 1 + 1\right) \\
&\leq 1_{\{|x|\leq \alpha+\frac{1}{n}\}}\|f\|^2_\infty \|g\|^2_\infty \left(2 + 2n^2 \left(\sum_j \frac{1}{K} 1_{\{U_j\leq K\}} 1_{\{\tau^\alpha_j<t\}}\right)^2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left[\int_0^t \int_{\mathbb{R}} 1_{\{|x|\leq \alpha+\frac{1}{n}\}}\|f\|^2_\infty \|g\|^2_\infty \left(2 + 2n^2 \left(\sum_j \frac{1}{K} 1_{\{U_j\leq K\}} 1_{\{\tau^\alpha_j<t\}}\right)^2\right)\right] dx\, ds \\
&= 2t\left(\alpha + \frac{1}{n}\right)\|f\|^2_\infty \|g\|^2_\infty \left(2 + 2n^2\, E\left(\sum_j \frac{1}{K} 1_{\{U_j\leq K\}} 1_{\{\tau^\alpha_j<t\}}\right)^2\right) \\
&= 2t\left(\alpha + \frac{1}{n}\right)\|f\|^2_\infty \|g\|^2_\infty (2 + 2n^2 c_2)
\end{aligned}
$$

the result follows from equation (3.38) and the dominated convergence theorem. ☐

Lemma 3.8.3.

$$
H_n \xrightarrow[n\to\infty]{L^1} H
$$

Proof. Fix $t > 0$ and $\omega \in \Omega$. Define

$$
\begin{aligned}
\mu^h_n(x) &:= n \int_x^{x+\frac{1}{n}} \int_E h(y, z)\hat{\nu}_t(y, dz)\, dy \\
\mu^h(x) &:= \int_E h(x, z)\hat{\nu}_t(x, dz)
\end{aligned}
$$

By the fundamental theorem of calculus, we have $\mu^{fg}_n \to \mu^{fg}$, and so $(\mu^{fg}_n)^2 \to (\mu^{fg})^2$, almost everywhere on $\mathbb{R}$. Noting that

$$
(\mu^{fg}_n(x))^2 \leq \|f\|^2_\infty \|g\|^2_\infty 1_{[-\alpha-\frac{1}{n}, \alpha+\frac{1}{n}]}(x)
$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ and that this bound is integrable over $\mathbb{R}$, we may apply the dominated convergence theorem to conclude that

$$\int_{\mathbb{R}} (\mu_n^{fg}(x))^2 dx \to \int_{\mathbb{R}} (\mu^{fg}(x))^2 dx \tag{3.39}$$

Also, observe that (by interchanging integrals) we have

$$\int_{\mathbb{R}} \mu_n^{f^2 g^2}(x) dx = \int_{\mathbb{R}} \mu^{f^2 g^2}(x) dx \tag{3.40}$$

Combining equations (3.39) and (3.40), we have

$$\left| \int_{\mathbb{R}} \left(\mu_n^{f^2 g^2}(x) - \mu^{f^2 g^2}(x) - (\mu_n^{fg}(x))^2 + (\mu^{fg}(x))^2 \right) dx \right| \to 0$$

for all $(s, \omega) \in [0, t] \times \Omega$, and as the left-hand side is bounded by

$$8 \left(\alpha + \frac{1}{n} \right) \|f\|_\infty^2 \|g\|_\infty^2 < \infty$$

the result follows by the usual bounded convergence theorem with respect to the integral $E \int_0^t \ldots ds$ □

This completes the proof of Theorem 3.6.2.

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