Spatial Moran Models with Local Interactions

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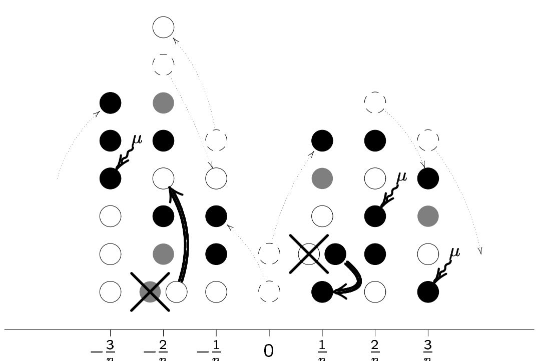
Chapter 1: Spatial Moran Models

- Construct a stepping stone Moran model;
- Derive limiting system of interacting Brownian motions.

Stepping Stone Model

- one-dimensional lattice $n^{-1}\mathbb{Z}$ of sites;
- initially distributed iid Poisson mean $n^{-1}K$ at each site;
- random walk migration (rate θn^2);
- mutation of types;
- Moran dynamics at each site.



The Generator

For $\mathbf{x} \in (n^{-1}\mathbb{Z})^\infty$ and $\mathbf{z} \in E^\infty$:

$$\tilde{A}_n f(\mathbf{x},\mathbf{z}) := \sum_j B^\theta_{n,j} f(\mathbf{x},\mathbf{z}) + \sum_j B^\mu_j f(\mathbf{x},\mathbf{z}) + \sum_{\substack{i\neq j\\ x_i = x_j}} \left(\lambda/2 + \sigma(z_i, z_j)\right)\left(f\big(\mathbf{x}, \eta_j(\mathbf{z}|z_i)\big) - f(\mathbf{x},\mathbf{z})\right)$$

with $\eta_j(\mathbf{z}|z_0) := (z_1, z_2, \ldots, z_{j-1}, z_0, z_{j+1}, \cdots)$ and

$$B^\theta_{n,j} f(\mathbf{x},\mathbf{z}) := \theta n^2 \Bigg(\frac{1}{2} f\big(\eta_j(\mathbf{x} \mid x_j + 1/n), \mathbf{z}\big) + \frac{1}{2} f\big(\eta_j(\mathbf{x} \mid x_j - 1/n), \mathbf{z}\big) - f(\mathbf{x},\mathbf{z})\Bigg)$$

and something like

$$B^\mu_j f(\mathbf{x},\mathbf{z}) := \int_E \big(f(\mathbf{x}, \eta_j(\mathbf{z}|z_0)) - f(\mathbf{x},\mathbf{z})\big)\mu(z_j, dz_0)$$

Explicit Construction

We will construct a solution $(\mathbf{X}^{(n)}, \tilde{\mathbf{Z}}^{(n)})$ as follows. Define location processes:

$$X_j^{(n)}(t) := X_j^{(n)}(0) + \frac{1}{n} W_j^{\mathbb{Z}}(\theta n^2 t)$$

and define pairwise interaction counting processes:

$$\tilde{V}_{ij}^{(n),\lambda}(t) := N_{ij}^{\lambda}\left(\frac{\lambda}{2}\int_0^t 1_{\{X_i^{(n)}(s)=X_j^{(n)}(s)\}} ds\right)$$
$$\bar{V}_{ij}^{(n),\sigma}(t) := \bar{N}_{ij}^{\sigma}\left(\bar{\sigma}\int_0^t 1_{\{X_i^{(n)}(s)=X_j^{(n)}(s)\}} ds\right)$$

to count when j (maybe) copies i's type.

(Where we've assumed $0 \leq \sigma(\cdot,\cdot) \leq \bar{\sigma}$.)

Defining the Type Process

Order all random times when j (maybe) copies someone's type:

$$0 \equiv \tilde{\tau}_{j,0} < \tilde{\tau}_{j,1} < \tilde{\tau}_{j,2} < \tilde{\tau}_{j,3} < \dots$$

Let $\tilde{Z}_j^{(n)}$ be such that

$$\tilde{Z}_j^{(n)}(t) = Y_{jk}\left(\tilde{Z}_j^{(n)}(\tilde{\tau}_{jk}), t - \tilde{\tau}_{jk}\right),$$
$$\tilde{\tau}_{jk} \leq t < \tilde{\tau}_{j,k+1},\ k \in \mathbb{Z}^+$$

where $Y_{jk}(y,\cdot)$ are independent copies of the mutation process started at y.

Now, we need only define all $\tilde{Z}_j^{(n)}(\tilde{\tau}_{jk})$.

Defining the Type Process (cont.)

For $k \in \mathbb{N}$, time $\tilde{\tau}_{jk}$ is an interaction of j with a specific particle i.

If potential selective, event becomes *actual* selective with probability:

$$\bar{\sigma}^{-1}\sigma\left(\tilde{Z}_i^{(n)}(\tilde{\tau}_{jk}-), \tilde{Z}_j^{(n)}(\tilde{\tau}_{jk}-)\right)$$

If neutral, event is *always* "actual."

Take new type of j to be:

$$\tilde{Z}_j^{(n)}(\tilde{\tau}_{jk}) = \begin{cases} \tilde{Z}_i^{(n)}(\tilde{\tau}_{jk}-), & \text{if event is actual;} \\ \tilde{Z}_j^{(n)}(\tilde{\tau}_{jk}-), & \text{otherwise.} \end{cases}$$

Potential vs. Actual

Potential selective events $\bar{V}_{ij}^{(n),\sigma}$ take place at time change

$$\bar{\sigma}\int_0^t 1_{\{X_i^{(n)}(s)=X_j^{(n)}(s)\}} ds$$

and are filtered according to

$$\bar{\sigma}^{-1}\sigma\left(\tilde{Z}_i^{(n)}(s-), \tilde{Z}_j^{(n)}(s-)\right)$$

so actual selective events $\tilde{V}_{ij}^{(n),\sigma}$ take place at time change

$$\int_0^t \sigma\left(\tilde{Z}_i^{(n)}(s), \tilde{Z}_j^{(n)}(s)\right) 1_{\{X_i^{(n)}(s)=X_j^{(n)}(s)\}} ds$$

Brownian Limit

As $n \to \infty$, location processes converge:

$$X_j^{(n)} \Rightarrow X_j := X_j(0) + \sqrt{\theta} W_j$$

and (scaled) interaction integrals converge:

$$n \int_0^{\cdot} 1_{\{X_i^{(n)}(s) = X_j^{(n)}(s)\}} ds \Rightarrow (2\theta)^{-1} L_t^0 (X_i - X_j)$$

Brownian Model

- particles live on $\mathbb{R}$;
- initially distributed $Poisson(K\ell_{\mathbb{R}})$;
- Brownian migration;
- mutation of types;
- pairwise Moran interactions driven by local times.

Explicit Construction

Define location processes:

$$X_j(t) := X_j(0) + \sqrt{\theta} W_j(t)$$

and define pairwise interaction counting processes:

$$\tilde{V}_{ij}^{\lambda}(t) := N_{ij}^{\lambda}\left(\frac{\lambda}{4\theta} L^0(X_i - X_j)\right)$$

$$\bar{V}_{ij}^{\sigma}(t) := \bar{N}_{ij}^{\sigma}\left(\frac{\bar{\sigma}}{2\theta} L^0(X_i - X_j)\right)$$

to count when j (maybe) copies i's type.

Create a type process as before:

- mutate between interactions $Y_{jk}(y, \cdot)$;
- at interaction, copy if "actual" event.

Chapter 2: Infinite-Density Stepping Stone Model

- Construct an ordered stepping stone model;
- Couple Moran and ordered models (generator argument);
- Construct an infinite-density neutral model embedding the finite-density neutral models.

Ordering the Model

Assign iid uniform levels to particles, independent of location and type.

Neutral interactions only occur in one direction. Only the higher-level particle changes its type.

Comparison of Generators

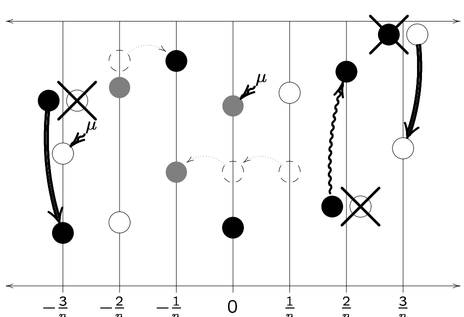
For $\mathbf{x} \in (n^{-1}\mathbb{Z})^\infty$ and $\mathbf{z} \in E^\infty$:

$$\tilde{A}_n f(\mathbf{x},\mathbf{z}) := \sum_j B^\theta_{n,j} f(\mathbf{x},\mathbf{z}) + \sum_j B^\mu_j f(\mathbf{x},\mathbf{z}) + \sum_{\substack{i\neq j\\ x_i=x_j}} \big(\lambda/2 + \sigma(z_i,z_j)\big)\big(f\big(\mathbf{x},\eta_j(\mathbf{z}|z_i)\big) - f(\mathbf{x},\mathbf{z})\big)$$

versus (for $\mathbf{u} \in [0,1]^\infty$)

$$A_n f(\mathbf{x},\mathbf{z},\mathbf{u}) := \sum_j B^\theta_{n,j} f(\mathbf{x},\mathbf{z},\mathbf{u}) + \sum_j B^\mu_j f(\mathbf{x},\mathbf{z},\mathbf{u}) + \sum_{\substack{i\neq j\\ x_i=x_j}} \big(1_{\{u_i<u_j\}}\lambda + \sigma(z_i,z_j)\big)\big(f\big(\mathbf{x},\eta_j(\mathbf{z}|z_i)\big) - f(\mathbf{x},\mathbf{z})\big)$$

Ordered Model in Action



Coupling via Generators

If $V^{(n),\lambda}_{ij}$ is the counting process of (ordered) neutral interactions:

$$V^{(n),\lambda}_{ij}(t) := 1_{\{U_i<U_j\}} N^\lambda_{ij}\left(\lambda \int_0^t 1_{\{X^{(n)}_i(s)=X^{(n)}_j(s)\}} ds\right)$$

then let Φ be such that

$$\Phi_j(t) = j + \sum_{i\neq j} \int_0^t \big(\Phi_i(s-) - \Phi_j(s-)\big) \widehat{V}_{\Phi_i(s-),\Phi_j(s-)}(ds)$$

where

$$\widehat{V}_{ij}(t) := \sum_{l=1}^{V^{(n),\lambda}_{ij}(t)} \xi_{ijl}$$

for ξ_{ijl} iid fair coin flips.

Note $j \mapsto \Phi_j(t)$ is a permutation of indices $\mathbb{N}$.

Coupling via Generators (cont.)

Using filtered martingale problem machinery, we show that

$$(\mathbf{X}_\Phi^{(n)}, \mathbf{Z}_\Phi^{(n)}) =^d (\mathbf{X}^{(n)}, \tilde{\mathbf{Z}}^{(n)})$$

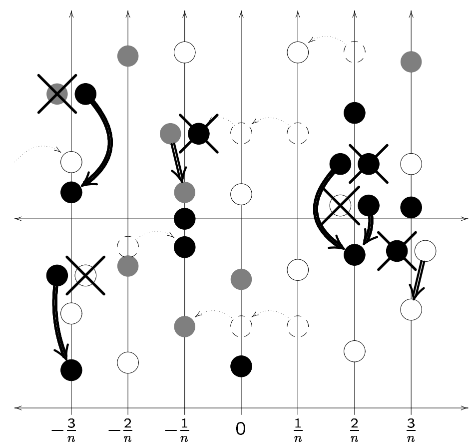
where $\mathbf{X}_\Phi^{(n)}$ means:

$$\left(X_{\Phi_1(\cdot)}^{(n)}(\cdot), X_{\Phi_2(\cdot)}^{(n)}(\cdot), \ldots\right)$$

Since Φ is permutation-valued, it follows that

$$\sum_j \delta_{(X_j^{(n)}(\cdot), \tilde{Z}_j^{(n)}(\cdot))} =^d \sum_j \delta_{(X_j^{(n)}(\cdot), Z_j^{(n)}(\cdot))}$$

Infinite-Density, Neutral Model



Chapter 3 (first half): Infinite-Density Brownian Model

- Construct a (finite-density) ordered system of interacting Brownian motions;
- Study selection mechanism in limit to construct an infinite-density ordered model *with selection*;
- Couple Moran and ordered models (level-flipping argument):
  - Couple finite-density Moran and ordered models;
  - Couple infinite-density ordered model to "hybrid" model.

Finite-Density Ordered Model

Define location processes:

$$X_j(t) := X_j(0) + \sqrt{\theta} W_j(t)$$

and define pairwise interaction counting processes:

$$V_{ij}^\lambda(t) := 1_{\{U_i < U_j\}} N_{ij}^\lambda\left(\frac{\lambda}{2\theta} L^0(X_i - X_j)\right)$$

$$\bar{V}_{ij}^\sigma(t) := \bar{N}_{ij}^\sigma\left(\frac{\bar{\sigma}}{2\theta} L^0(X_i - X_j)\right)$$

to count when j (maybe) copies i's type.

Create a type process as before:

- mutate between interactions $Y_{jk}(y, \cdot)$;
- at interaction, copy if "actual" event.

Limiting Selection

Define

$$\bar{V}_j^\sigma := \sum_{i \neq j} \bar{V}_{ij}^\sigma$$

counting *all* potential selective events affecting particle j.

Then

$$\bar{V}_j^\sigma = \bar{N}_j^\sigma \left(\frac{\bar{\sigma}}{2\theta} \sum_{i \neq j} L_{ij}(\cdot) \right)$$

and, for each fixed j,

$$\frac{\bar{\sigma}_0}{2\theta K} \sum_{i \neq j} L_{ij}(t) \xrightarrow[K \to \infty]{L^2} \bar{\sigma}_0 t$$

Infinite-Density Ordered Model

Define location processes:

$$X_j(t) := X_j(0) + \sqrt{\theta} W_j(t)$$

Define pairwise neutral counting processes:

$$V_{ij}^\lambda(t) := 1_{\{U_i < U_j\}} N_{ij}^\lambda \left(\frac{\lambda}{2\theta} L^0(X_i - X_j) \right)$$

to count when j copies i's type

Define per-particle potential selective event counting processes:

$$\bar{V}_j^\sigma(t) := \bar{N}_j^\sigma(\bar{\sigma} t)$$

to count when j (maybe) copies the type of a "nearby" particle.

Selective Events

Create a type process *almost* as before:

- mutate between interactions $Y_{jk}(y, \cdot)$;
- at interaction, copy if "actual" event.

But, *what* do we copy for a selective event?

For now, suppose

$$Z_0 = \psi \left(\sum_i \delta_{(X_i(s-), Z_i(s-), U_i)}, X_j(s-), \eta \right)$$

gives us a candidate type when $\eta \sim U[0,1]$. As before, potential becomes actual with probability

$$\bar{\sigma}^{-1} \sigma(Z_0, Z_j(s-))$$

Coupling via Level-Flipping

Create a finite-density intermediate model:

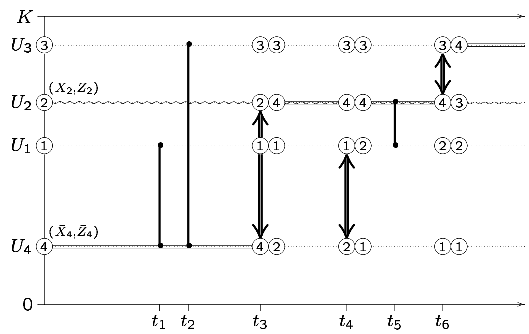
- particles start with iid uniform levels;
- neutral events are ordered (high copies low);
- immediately after a neutral event, particles swap levels half the time.

Ignore levels and follow indices, and you "see" the symmetric model.

Ignore indices and follow levels, and you "see" the ordered model.

Intermediate Model

ⓙ = level of particle j



Consequence: $\sum_j \delta_{(\tilde{X}_j(\cdot),\tilde{Z}_j(\cdot))} = \sum_j \delta_{(X_j(\cdot),Z_j(\cdot))}$

Coupling via Level-Flipping (cont.)

Also can create an infinite-density hybrid model:

- below level K, looks like symmetric;
- above level K, looks like ordered.

As a consequence, the infinite-density model can be broken in two at any level, and you can "ignore the levels" of the bottom part.

Chapter 3 (second half): Infinite-Density Brownian Model

- Define infinite-density location/type measure-valued process (via backwards martingales "in level space");
- Establish conditional Poisson structure of particle system;
- Characterize the measure-valued process with martingales using:
  - "hybrid" model symmetry;
  - backwards martingale convergence;
  - Poisson structure.

Location/Type Measure

Write

$$\left\langle u_t^K, h \right\rangle := \sum_{U_j \leq K} h\left(X_j(t), Z_j(t)\right)$$

Note that $\frac{1}{K}\left\langle u_t^K, h\right\rangle$ is the average "h-ness" of all particles below level K.

If limits

$$\langle u_t, h \rangle := \lim_{K\to\infty} \frac{1}{K}\left\langle u_t^K, h\right\rangle$$

exist, they will characterize the location/type distribution of the infinite-density process.

Location/Type Measure (cont.)

Define filtrations

$$\mathfrak{F}_t^K := \sigma\Big\{\sum_{U_j \le K} \delta_{(X_j(r), Z_j(r))}, \sum_{U_j > K} \delta_{(X_j(r), Z_j(r), U_j)}, r \le t\Big\}$$

and $\mathfrak{F}_t^\infty = \bigcap_{K>0} \mathfrak{F}_t^K$.

By the hybrid model coupling:

$$\mathsf{E}\left[\left\langle u_t^1, h\right\rangle \,\middle|\, \mathfrak{F}_t^K\right] = \frac{1}{K}\left\langle u_t^K, h\right\rangle$$

and by backwards martingale convergence:

$$\mathsf{E}\left[\left\langle u_t^1, h\right\rangle \,\middle|\, \mathfrak{F}_t^K\right] \underset{K\to\infty}{\longrightarrow} \mathsf{E}\left[\left\langle u_t^1, h\right\rangle \,\middle|\, \mathfrak{F}_t^\infty\right]$$

Poisson Structure

Assume Poisson starting conditions.

For every $t \ge 0$, conditioned on $\mathfrak{F}_t^\infty$,

$$\xi_t := \sum_j \delta_{(X_j(t), Z_j(t), U_j)} \sim Poisson(\nu_t \times \ell_{\mathbb{R}+})$$

where

$$\nu_t(A \times B) := \langle u_t, 1_{A\times B}\rangle$$

Also, $\nu_t(\cdot \times E) = \ell_{\mathbb{R}}$.

Selection Revisited

There exists $\widehat{\nu}$ such that

$$\nu_t(C) = \int_{\mathbb{R}}\int_E 1_C(x,z)\widehat{\nu}_t(x,dz)dx$$

for all $C \in \mathfrak{B}(\mathbb{R} \times E)$.

So, $\widehat{\nu}_t(x,\cdot)$ is rather like the probability distribution of types "at" point x.

Wouldn't it be nice if, conditioned on $\mathfrak{F}_t^{\mathbf{X},\mathbf{Z},\mathbf{U}}$, the candidate type

$$Z_0 = \psi\left(\xi_{s-}, X_j(s-), \eta\right)$$

had distribution $\widehat{\nu}_t(X_j(s-),\cdot)$?

Martingale Characterization

All this machinery allows us to show that, for $f \in C_c^2(\mathbb{R})$ and $g \in \mathfrak{D}(B^\mu)$,

$$M_t := \langle u_t, fg\rangle - \langle u_0, fg\rangle - \int_0^t \left\langle u_s, \tfrac{\theta}{2} f'' g + f B^\mu g + h_s^\sigma \right\rangle ds$$

is an $\{\mathfrak{F}_t^\infty\}$-martingale where

$$h_s^\sigma(x,z,\omega) := f(x)\int_0^1 \sigma\Big(\psi(\xi_{s-}(\omega), x, \eta), z\Big)\Big(g\Big(\psi(\xi_{s-}(\omega), x, \eta)\Big) - g\Big(z\Big)\Big)d\eta$$

Quadratic Variation

Using the same machinery, we can establish that

$$\left\langle M^K \right\rangle_t \xrightarrow[K\to\infty]{L^1} \langle M \rangle_t$$

where

$$M_t^K := \frac{1}{K}\Bigg(\left\langle u_t^K, fg \right\rangle - \left\langle u_0^K, fg \right\rangle - \int_0^t \left\langle u_s^K, \tfrac{\theta}{2} f'' g + f B^\mu g + h_s^\sigma \right\rangle ds\Bigg)$$

Quadratic Variation (cont.)

Using a physically painful analysis argument, we can establish that

$$\langle M \rangle = \theta\lambda \int_0^{\cdot} ds \int_{\mathbb{R}} dx \int_{E\times E} \widehat{\nu}_s(x, dz)\widehat{\nu}_s(x, dz') \Big(f(x)g(z) - f(x)g(z')\Big)^2$$

Tightness

Using another involved argument, we can establish that $\left(\frac{1}{K}\left\langle u_\cdot^K, fg \right\rangle\right)_K$ and $\left(M^K\right)_K$ are tight.

A consequence of this, since the maximum jump size goes to zero, is that the limits are pathwise continuous. (And the angle-brackets process of M is also its quadratic variation.)

Complete Martingale Problem

For $f \in C_c^2(\mathbb{R})$ and $g \in \mathfrak{D}(B^\mu)$,

$$M_t := \langle u_t, fg \rangle - \langle u_0, fg \rangle - \int_0^t \left\langle u_s, \tfrac{\theta}{2} f'' g + f B^\mu g + h_s^\sigma \right\rangle ds$$

is an $\{\mathfrak{F}_t^\infty\}$-martingale with quadratic variation

$$[M^K] = \theta\lambda \int_0^{\cdot} ds \int_{\mathbb{R}} dx \int_{E\times E} \widehat{\nu}_s(x, dz)\widehat{\nu}_s(x, dz') \Big(f(x)g(z) - f(x)g(z')\Big)^2$$

Mueller-Tribe SPDE

- Two types $E = \{0,1\}$;
- No mutation;
- $\sigma(1,0) = \bar{\sigma}$.

Then, for $g = \delta_1$ and $u(s,x) = \hat{\nu}_s(x, \delta_1)$, we have

$$\int_{\mathbb{R}} f(x)u(t,x)dx - \int_{\mathbb{R}} f(x)u(0,x)dx - \int_0^t \int_{\mathbb{R}} \Big(\frac{\theta}{2} f''(x)u(s,x) + \bar{\sigma} u(s,x)(1-u(s,x)) \Big) dx\, ds$$

a martingale with quadratic variation

$$2\theta\lambda \int_0^t \int_{\mathbb{R}} f^2(x)u(s,x)(1-u(s,x))dx\, ds$$

Thus, u is a weak solution of:

$$\dot{u} = \frac{\theta}{2}\Delta u + \bar{\sigma} u(1-u) + \sqrt{2\theta\lambda u(1-u)}\dot{W}$$