A Brownian Particle System with Local Time Interaction

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December 6, 2001

Symmetric Particle Model

Consider a K-density collection of particles located in $\mathbb{R}$ with types in E.

Start them at:

$$\sum \delta_{(X_j(0),Z_j(0))} \sim PPP(K\nu)$$

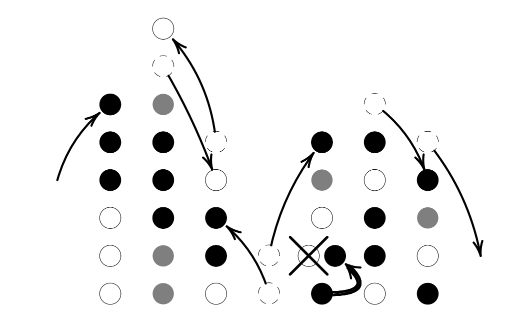
where ν is a measure on $\mathbb{R} \times E$ with $\nu(\cdot, E) = \ell_{\mathbb{R}}$.

Evolve them as:

$$
\begin{aligned}
&\text{(Locations)} & X_j(t) &= X_j(0) + \sqrt{\theta} W_j(t) \\
&\text{(Interactions)} & \tilde{V}_{ij}(t) &= N_{ij}\left(\tfrac{\lambda}{2} L_t^0(X_i - X_j)\right) \\
&\text{(Types)} & \tilde{Z}_j(t) &= Z_j(0) \\
& & &\quad + \sum_{i \neq j} \int_0^t (\tilde{Z}_i(s-) - \tilde{Z}_j(s-)) d\tilde{V}_{ij}(s) \\
& & &\quad + Mutation(t)
\end{aligned}
$$

Genetic Stepping-Stone Model

- one-dimensional lattice $n^{-1}\mathbb{Z}$ of sites;
- initially distributed iid Poisson mean $n^{-1}K$;
- site-to-site migration (rate θn^2);
- Moran dynamics (rate n) at each site.



Genetic Stepping-Stone Model

Consider a K-density collection of particles living in $n^{-1}\mathbb{Z}$ with types in E.

Start them at:

$$\sum \delta_{(X_j^n(0),Z_j^n(0))} \sim PPP(K\nu^n)$$

where ν^n is a measure on $n^{-1}\mathbb{Z} \times E$ with $\nu^n(\cdot, E) = \sum_{x \in n^{-1}\mathbb{Z}} n^{-1}\delta_x$.

Evolve them as:

$$
\begin{aligned}
&\text{(Locations)} & X_j^n(t) &= X_j^n(0) + \tfrac{1}{n} W_j^{\mathbb{Z}}(\theta n^2 t) \\
&\text{(Interactions)} & \tilde{V}_{ij}^n(t) &= N_{ij}\left(\tfrac{\lambda}{2} \int_0^t n \cdot 1_{\{X_i^n(s)=X_j^n(s)\}} 2\theta ds\right) \\
&\text{(Types)} & \tilde{Z}_j^n(t) &= Z_j^n(0) \\
& & &\quad + \sum_{i \neq j} \int_0^t (\tilde{Z}_i^n(s-) - \tilde{Z}_j^n(s-)) d\tilde{V}_{ij}^n(s) \\
& & &\quad + Mutation(t)
\end{aligned}
$$

Stepping-Stone Limit

We have

$$((X_j^n)_j, (\tilde{V}_{ij}^n)_{i \neq j}) \Rightarrow ((X_j)_j, (\tilde{V}_{ij})_{i \neq j})$$

in J_1 topology.

Under $\nu^n \to \nu$ and some regularity conditions on mutation, we will have

$$(X_j^n, \tilde{Z}_j^n)_j \Rightarrow (X_j, \tilde{Z}_j)_j$$

Brownian Particle System

Instead of

$$\tilde{V}_{ij}(t) = N_{ij}\left(\frac{\lambda}{2} L_t^0 (X_i - X_j)\right)$$

define U_j iid $U[0, K]$ and take

$$V_{ij}(t) = N_{ij}\left(1_{\{U_i < U_j\}} \lambda L_t^0 (X_i - X_j)\right)$$

Types evolve as:

$$Z_j(t) = Z_j(0) + \sum_{i \neq j} \int_0^t (Z_i(s-) - Z_j(s-)) dV_{ij}(s) + Mutation(t)$$

Infinite-Density System ξ

Start particles at:

$$\sum \delta_{(X_j(0), Z_j(0), U_j)} \sim PPP(\nu \times \ell_{\mathbb{R}+})$$

Evolve them as:

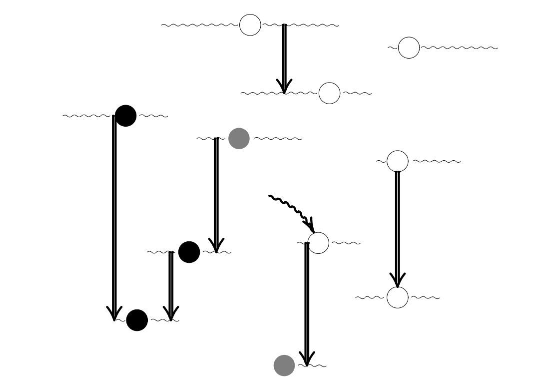
(Locations) $$X_j(t) = X_j(0) + \sqrt{\theta} W_j(t)$$

(Interactions) $$V_{ij}(t) = N_{ij}\left(1_{\{U_i < U_j\}} \lambda L_t^0 (X_i - X_j)\right)$$

(Types) $$Z_j(t) = Z_j(0) + \sum_{i \neq j} \int_0^t (Z_i(s-) - Z_j(s-)) dV_{ij}(s) + Mutation(t)$$

Write $\xi_t = \sum_j \delta_{(X_j(t), Z_j(t), U_j)}$.

Infinite-Density System ξ



An Intermediate Model

LTFBI:

- $\sum_j \delta_{(X_j(0),Z_j(0))} \sim PPP(K\nu)$;
- W_j iid Brownian motions on $\mathbb{R}$;
- $N_{\{i,j\}}$ iid rate 1 Poisson counters;
- $\pi_{\{i,j\},k}$ iid uniform random $\{i,j\} \leftrightarrow \{i,j\}$;
- U_j iid $U[0,K]$.

Define Brownian locations

$$\tilde{X}_j(t) = X_j(0) + \sqrt{\theta} W_j(t)$$

and directionless interactions

$$V_{\{i,j\}}(t) = N_{\{i,j\}}(\lambda L_t^0(\tilde{X}_i - \tilde{X}_j))$$

Ordering the Model

Some notation:

- Let $\{T_{\{i,j\},k}\}_{k\in\mathbb{N}}$ be the ordered jump times of $V_{\{i,j\}}$;
- Let $\{(T_{j,k}, \pi_{j,k})\}$ be the strict ordering of $\{(T_{\{i,j\},k}, \pi_{\{i,j\},k}) : \forall i \neq j, \forall k\}$ by first component.

That is, j interacts at $T_{j,1} < T_{j,2} < \ldots$, and at $T_{j,k}$, there is a $\pi_{j,k}$: $\{i,j\} \leftrightarrow \{i,j\}$.

Ordering the Model

Theorem 1. *There exist* $[0,K]$*-valued, left-continuous, jump processes* $\tilde{U}_j$ *satisfying*

$$\tilde{U}_j(t) = \begin{cases} U_j, & 0 \le t \le T_{j,1}; \\ \tilde{U}_{\pi_{j,l}(j)}(T_{j,l}), & T_{j,l} < t \le T_{j,l+1}. \end{cases}$$

and $\mathbb{N}$*-valued, left-continuous jump processes* Φ_j *satisfying*

$$\tilde{U}_{\Phi_j(t)}(t) = U_j$$

For each fixed $t \ge 0$*, conditioned on*

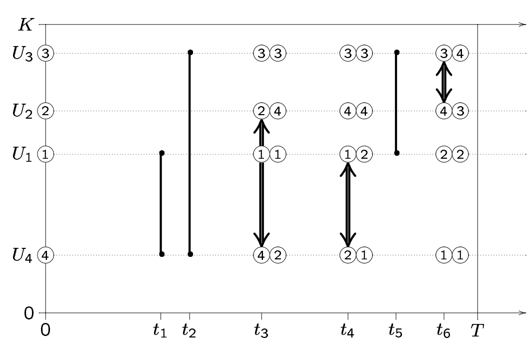
$$\sigma\left\{T_{\{i,j\},k}, 1_{\{\tilde{U}_i(T_{\{i,j\},k}) < \tilde{U}_j(T_{\{i,j\},k})\}} : T_{\{i,j\},k} < t\right\}$$

the $\tilde{U}_j(t)$ *are iid* $U[0,K]$.

(Thus, conditioned on $\sigma\{T_{\{i,j\},k}\}$*, the* $1_{\{\tilde{U}_i(T_{\{i,j\},k}) < \tilde{U}_j(T_{\{i,j\},k})\}}$ *are iid coin flips.)*

Ordered Intermediate Model

ⓙ = level of particle j



j	$\Phi_j(0+)$	$\Phi_j(t_3+)$	$\Phi_j(t_4+)$	$\Phi_j(t_6+)$	$\tilde{U}_{\Phi_j(t)}(t)$
1	1	1	2	2	U_1
2	2	4	4	3	U_2
3	3	3	3	4	U_3
4	4	2	1	1	U_4

Deriving the Symmetric Particle Model

Define $\tilde{V}_{ij}$ (started at 0) by

$$\Delta \tilde{V}_{ij}(t) = \sum_{k=1}^{\infty} 1_{\{t=T_{\{i,j\},k}\}} 1_{\{\tilde{U}_i(t)<\tilde{U}_j(t)\}}$$

By Theorem 1,

$$\tilde{V}_{ij} =^d N_{ij}\left(\tfrac{\lambda}{2} L_t^0(\tilde{X}_i - \tilde{X}_j)\right)$$

If we define

$$\begin{aligned}\tilde{Z}_j(t) &= Z_j(0) \\ &\quad + \sum_{i\neq j} \int_0^t (\tilde{Z}_i(s-) - \tilde{Z}_j(s-)) d\tilde{V}_{ij}(s) \\ &\quad + Mutation(t)\end{aligned}$$

then $(\tilde{X}_j, \tilde{Z}_j)$ is the K-density symmetric model.

Deriving the Infinite-Density System ξ

Define Brownian locations

$$X_j(t) = \tilde{X}_{\Phi_j(t)}(t)$$

Define V_{ij} (started at 0) by

$$\Delta V_{ij}(t) = 1_{\{U_i<U_j\}} \sum_{k=1}^{\infty} 1_{\{t=T_{\{\Phi_i(t),\Phi_j(t)\},k}\}}$$

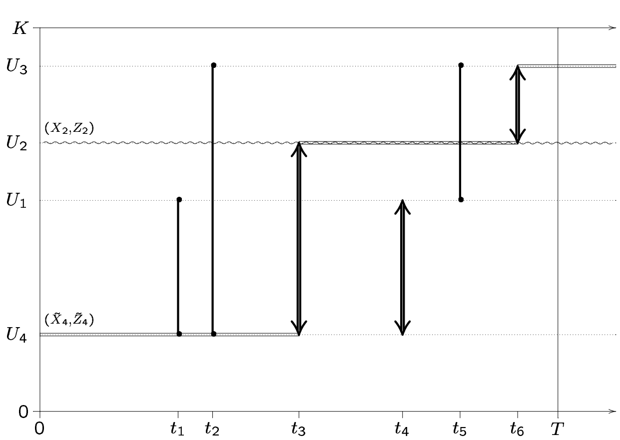
If we define

$$\begin{aligned}Z_j(t) &= Z_j(0) \\ &\quad + \sum_{i\neq j} \int_0^t (Z_i(s-) - Z_j(s-)) dV_{ij}(s) \\ &\quad + Mutation(t)\end{aligned}$$

then

$$\sum \delta_{(X_j,Z_j,U_j)} =^d \xi|_{\mathbb{R}\times E\times[0,K]}$$

Symmetric and ξ Particles



Consequence: $\sum \delta_{(\tilde{X}_j,\tilde{Z}_j)} = \sum \delta_{(X_j,Z_j)}$

Some Notation

Consider the infinite density model on $\mathbb{R}\times E\times\mathbb{R}^+$:

$$\xi_t = \sum_j \delta_{(X_j(t),Z_j(t),U_j)}$$

Write

$$\langle u_t^K, g\rangle = \sum_{U_j\le K} g(X_j(t), Z_j(t))$$

and define

$$\mathfrak{F}_t^K = \sigma\left\{\xi\Big|_{\mathbb{R}\times E\times(K,\infty)}(r), \sum_{U_j\le K} \delta_{(X_j,Z_j)}(r), r\le t\right\}$$

$$\mathfrak{F}_t^\infty = \bigcap_{K>0} \mathfrak{F}_t^K$$

Limiting Location-Type Measure

We are interested in $\langle u_t, g\rangle \equiv \lim_{K\to\infty} \frac{1}{K}\langle u_t^K, g\rangle$.

Theorem 2. *For $g \in B_c(\mathbb{R}\times E)$, we have*
$\frac{1}{K}\langle u_t^K, g\rangle = \mathsf{E}\Big[\langle u_t^1, g\rangle \;\Big|\; \mathfrak{F}_t^K\Big] \underset{K\to\infty}{\longrightarrow} \mathsf{E}\Big[\langle u_t^1, g\rangle \;\Big|\; \mathfrak{F}_t^\infty\Big]$
almost surely and in L_1.

Proof. By the coupling,

$$\begin{aligned}
&\mathsf{E}\Big[\langle u_t^1, g\rangle \;\Big|\; \mathfrak{F}_t^K\Big] \\
&\quad = \mathsf{E}\left[\sum_{U_j\le K} 1_{\{U_j\le 1\}} g(X_j(t), Z_j(t)) \;\middle|\; \mathfrak{F}_t^K\right] \\
&\quad = \mathsf{E}\left[\frac{1}{K}\sum_{U_j\le K} g(X_j(t), Z_j(t)) \;\middle|\; \mathfrak{F}_t^K\right]
\end{aligned}$$

Convergence follows as $\mathsf{E}\Big[\langle u_t^1, g\rangle \;\Big|\; \mathfrak{F}_t^K\Big]$ is a backwards martingale in K. □

Poisson Structure

Why is $\langle u_t, g\rangle \equiv \lim_{K\to\infty} \frac{1}{K}\langle u_t^K, g\rangle$ interesting?

Theorem 3. *For all $t \ge 0$, conditioned on $\mathfrak{F}_t^\infty$, the process ξ_t is $PPP(\nu_t \times \ell_{\mathbb{R}+})$ where $\nu_t(F) = \langle u_t, 1_F\rangle$.*

Remark 1. Note that $\nu_t(\cdot, E) = \ell_{\mathbb{R}}$. By Morando's Theorem, we may (with abuse of notation) write:

$$\nu_t(dx, dz) = \nu_t(x, dz)dx$$

An Itô Identity

Take the mutation to satisfy

$$g(Y(t)) = g(y) + \int_0^t Bg(Y(s))ds + M^g(t)$$

For $f \in C_c^2(\mathbb{R})$ and $g \in \mathfrak{D}(B)$, we claim

$$\begin{aligned}
&\langle u_t^1, fg\rangle - \langle u_0^1, fg\rangle - \int_0^t \Big\langle u_s^1, \tfrac{\theta}{2}f''g + fBg\Big\rangle\, ds \\
&= \sqrt{\theta}\sum_{U_j\le 1}\int_0^t f'(X_j(s))g(Z_j(s-))dW_j(s) \\
&\quad + \sum_{U_j\le 1}\int_0^t f(X_j(s))dM_j^g(s) \\
&\quad + \sum_{U_i<U_j\le 1}\int_0^t (f(X_i(s))g(Z_i(s-)) - \\
&\qquad\qquad f(X_j(s))g(Z_j(s-)))dV_{ij}(s)
\end{aligned}$$

is an $(\mathfrak{F}_t^1)$-martingale.

A Martingale Problem

Lemma 4. *If $\mathfrak{F}_t \subseteq \mathfrak{G}_t$ then*

$$H(t) - \int_0^t K(s)ds$$

a $(\mathfrak{G}_t)$-martingale implies

$$\mathsf{E}[H(t)|\mathfrak{F}_t] - \int_0^t \mathsf{E}[K(s)|\mathfrak{F}_s]ds$$

an $(\mathfrak{F}_t)$-martingale.

Therefore,

$$\langle u_t^1, fg\rangle - \langle u_0^1, fg\rangle - \int_0^t \Big\langle u_s^1, \tfrac{\theta}{2}f''g + fBg\Big\rangle\, ds$$

an $(\mathfrak{F}_t^1)$-martingale implies

$$\langle u_t, fg\rangle - \langle u_0, fg\rangle - \int_0^t \Big\langle u_s, \tfrac{\theta}{2}f''g + fBg\Big\rangle\, ds$$

is an $(\mathfrak{F}_t^\infty)$-martingale.

The Angle-Brackets Process $\langle M\rangle$

Consider the $(\mathfrak{F}_t^\infty)$- and $(\mathfrak{F}_t^K)$-martingales:

$$M_t = \langle u_t, fg\rangle - \langle u_0, fg\rangle - \int_0^t \left\langle u_s, \tfrac{\theta}{2} f'' g + fBg\right\rangle ds$$

$$M_t^K = \frac{1}{K}\left\{\langle u_t^K, fg\rangle - \langle u_0^K, fg\rangle - \int_0^t \left\langle u_s^K, \tfrac{\theta}{2} f'' g + fBg\right\rangle ds\right\}$$

Lemma 5. *For all $t \geq 0$, $M_t^K \xrightarrow{L_2} M_t$.*

Lemma 6. *If $\langle M^K\rangle_t \xrightarrow{L_1} A_t$ for predictable A, then $\langle M\rangle = A$.*

Proof. For $0 \leq s < t$,

$$0 = \mathsf{E}\Big[\Big((M_u^K)^2 - \langle M^K\rangle_u\Big)\Big|_{u=s}^{u=t} \;\Big|\; \mathfrak{F}_s^\infty\Big]$$
$$\xrightarrow{L_1} \mathsf{E}\Big[\Big((M_u)^2 - A_u\Big)\Big|_{u=s}^{u=t} \;\Big|\; \mathfrak{F}_s^\infty\Big]$$

□

Calculating $\langle M\rangle$

By the same Itô identity,

$$M_t^K = \frac{1}{K}\left\{\langle u_t^K, fg\rangle - \langle u_0^K, fg\rangle - \int_0^t \left\langle u_s^K, \tfrac{\theta}{2} f'' g + fBg\right\rangle ds\right\}$$
$$= \frac{1}{K}\Bigg\{\sqrt{\theta}\sum_{U_j\leq K}\int_0^t f'(X_j(s))g(Z_j(s-))dW_j(s)$$
$$+ \sum_{U_j\leq K}\int_0^t f(X_j(s))dM_j^g(s)$$
$$+ \sum_{U_i<U_j\leq K}\int_0^t (f(X_i(s))g(Z_i(s-))$$
$$- f(X_j(s))g(Z_j(s-)))dV_{ij}(s)\Bigg\}$$

Thus,

$$\langle M^K\rangle_t = O_p\left(\tfrac{1}{K}\right)$$
$$+ \frac{\lambda}{K^2}\sum_{U_i<U_j\leq K}\int_0^t (f(X_i(s))g(Z_i(s-))$$
$$- f(X_j(s))g(Z_j(s-)))^2 dL_t^0(X_i - X_j)(s)$$

Calculating $\langle M\rangle$

Write

$$h_{ij} = (f(X_i(s))g(Z_i(s-)) - f(X_j(s))g(Z_j(s-)))^2$$

Then

$$\frac{\lambda}{K^2}\sum_{U_i<U_j\leq K}\int_0^t h_{ij}\, dL_t^0(X_i - X_j)(s)$$
$$\approx \frac{\lambda}{K^2}\sum_{U_i<U_j\leq K}\int_0^t h_{ij}\int_{\mathbb{R}} n^2 1_{\{X_i(s),X_j(s)\in(x,x+\frac{1}{n})\}} dx\, 2\theta ds$$
$$\xrightarrow{L_1} 2\theta\lambda\int_0^t\int_{\mathbb{R}}\Bigg(n\int_x^{x+\frac{1}{n}}\int_E f^2(y)g^2(z)\nu_s(y,dz)dy$$
$$- \left(n\int_x^{x+\frac{1}{n}}\int_E f(y)g(z)\nu_s(y,dz)dy\right)^2\Bigg)dx\, ds$$
$$\approx 2\theta\lambda\int_0^t\int_{\mathbb{R}}\Bigg(\int_E f^2(x)g^2(z)\nu_s(x,dz)$$
$$- \left(\int_E f(x)g(z)\nu_s(x,dz)\right)^2\Bigg)dx\, ds$$
$$= \theta\lambda\int_0^t\int_{\mathbb{R}}\int_{E\times E} f^2(x)(g(z)-g(z'))^2\,\nu_s(x,dz)\nu_s(x,dz')dx\, ds$$

Complete Martingale Problem

For $f \in C_c^2(\mathbb{R})$ and $g \in \mathfrak{D}(B)$, we have

$$\langle u_t, fg\rangle - \langle u_0, fg\rangle - \int_0^t \left\langle u_s, \frac{\theta}{2} f'' g + fBg\right\rangle ds$$

an $(\mathfrak{F}_t^\infty)$-martingale with angle-brackets process (and quadratic variation)

$$\theta\lambda\int_0^t\int_{\mathbb{R}}\int_{E\times E} f^2(x)(g(z)-g(z'))^2\,\nu_s(x,dz)\nu_s(x,dz')dx\, ds$$

Is the solution unique?

Mueller-Tribe SPDE

- Two types $E = \{0, 1\}$;
- No mutation.

Then, for $g = \delta_1$ and $u(s, x) = \nu_s(x, \delta_1)$, we have

$$\int_{\mathbb{R}} f(x)u(t,x)dx - \int_{\mathbb{R}} f(x)u(0,x)dx - \frac{\theta}{2}\int_0^t f''(x)u(s,x)dx\,ds$$

a martingale with quadratic variation

$$2\theta\lambda \int_0^t\int_{\mathbb{R}} f^2(x)u(s,x)(1-u(s,x))dx\,ds$$

Thus, u is a weak solution of:

$$\dot{u} = \frac{\theta}{2}\Delta u + \sqrt{2\theta\lambda u(1-u)}\dot{W}$$