Interface Solutions of Lookdown Particle Systems, Part III: Interfaces of Fast-Interaction Models

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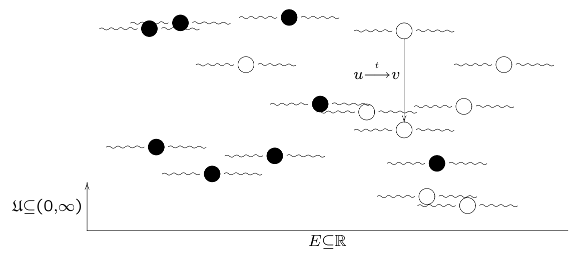
Lookdown Particle Systems



We define the particle system

$$\Psi_t := \sum_{u\in\mathfrak{U}} \delta_{(\widehat{Z}_u(t),\kappa_u(t),u)}$$
$$\in \mathfrak{M}_a(E \times \{0 \equiv \bigcirc, 1 \equiv \bullet\} \times (0,\infty))$$

where

- $\Psi_0 \sim \text{Poisson}(X_0 \times \ell_{(0,\infty)})$ for $X_0(\cdot \times \{0,1\}) = m$;
- $\widehat{Z}_u|\Psi_0$ iid copies of $\widehat{Z}$ (with stationary measure m);
- κ_u are determined by $\mathfrak{G} = \{u \xrightarrow{t} v\}$ such that

$$\kappa_u(t) = \begin{cases} \kappa_v(t-) & u \xrightarrow{t} v \\ \kappa_u(t-) & \text{otherwise} \end{cases}$$

Particle Motion with Dual

Continuous Markov motion $(\widehat{Z}, \widehat{P}_t)$ on interval $E \subseteq \mathbb{R}$ with dual (Z, P_t):

$$\int f P_t g \, dm = \int g \widehat{P}_t f \, dm$$

with diffuse m satisfying $0 < m(a,b) < \infty$ for all $a < b,\ a, b \in E$.

Hypothesis 1. There exists $E_n \to E$ relatively open in E with $m(E_n) < \infty$ and

$$\int_E \widehat{\mathsf{P}}^z\{\sigma_{E_n} \le t\} m(dz) < \infty$$

where

$$\sigma_A := \inf\{t \ge 0 : \widehat{Z}(t) \in A\}$$

Particle Systems with Fast Local Interactions (DEFKZ, 2000)

Let

$$[u]_t := \{v \in \mathfrak{U} : \widehat{Z}_u(t) = \widehat{Z}_v(t)\}$$

It turns out $[u]_t$ always has a minimum which we denote by $\lfloor u \rfloor_t$, and define

$$\mathfrak{G}^\infty = \{(t, u, \lfloor u \rfloor_t) : u \ne \lfloor u \rfloor_t,\ u \in \mathfrak{U}, t > 0\}$$

We showed (Dec 3) that there exist càdlàg κ_u satisfying this genealogy and so having

$$\kappa_u(t) = \kappa_{\lfloor u \rfloor_t}(t) = \kappa_{\lfloor u \rfloor_t}(t-)$$

Limiting Measure-Valued Process

Define a kind of M-average empirical measure:

$$X_t^M := \frac{1}{M} \sum_{u \in \mathfrak{U} \cap (0,M)} \delta_{(\widehat{Z}_u(t), \kappa_u(t))}$$

We showed (Feb 4) that for all $h \in B(E \times \{0,1\})$ with compact support,

$$X_t^M(h) \xrightarrow[M\to\infty]{a.s.,\, L_p} X_t(h)$$

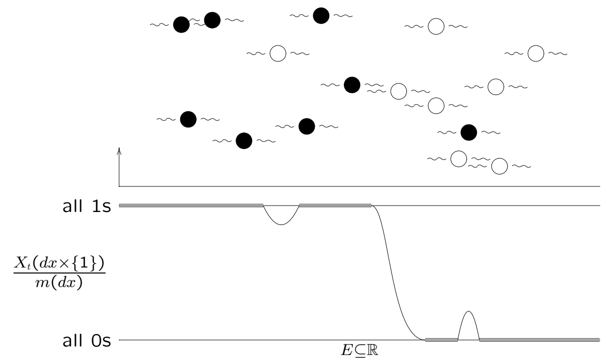
and $\exists$ filtration $\mathfrak{F}_t$ so that $X_t \in \mathfrak{F}_t$ and

$$\Psi_t \Big| \mathfrak{F}_t \sim \text{Poisson}(X_t \times \ell_{(0,\infty)})$$

where $X_t(\cdot \times \{0,1\}) = m$.

Particle System Carries Measure-Valued Process

So, Ψ_t is a particle construction of a measure-valued "process" X_t.



Note that:

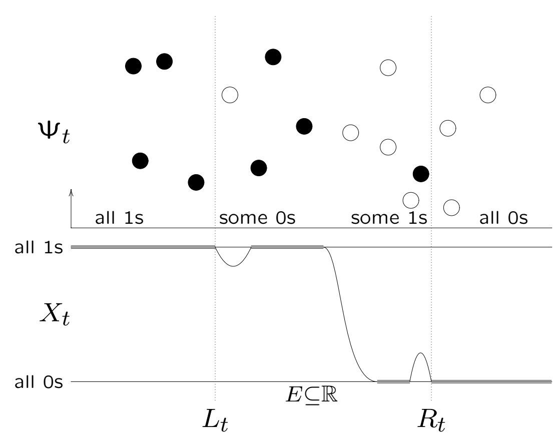
- $\Psi_0|\mathfrak{F}_0 \sim \text{Poisson}(X_0 \times \ell_{(0,\infty)})$ by construction;
- $\Psi_t|\mathfrak{F}_t \sim \text{Poisson}(X_t \times \ell_{(0,\infty)})$ by the previous slide.

An Interface of Ψ_t

For $\tilde{E}$ the closure of E in $\mathbb{R}_\star = [-\infty, +\infty]$, define $\tilde{E}$-valued processes

$$L_t := \inf\{\widehat{Z}_u(t) : \kappa_u(t) = 0\} \quad (= \sup E \text{ if } \emptyset)$$
$$R_t := \sup\{\widehat{Z}_u(t) : \kappa_u(t) = 1\} \quad (= \inf E \text{ if } \emptyset)$$

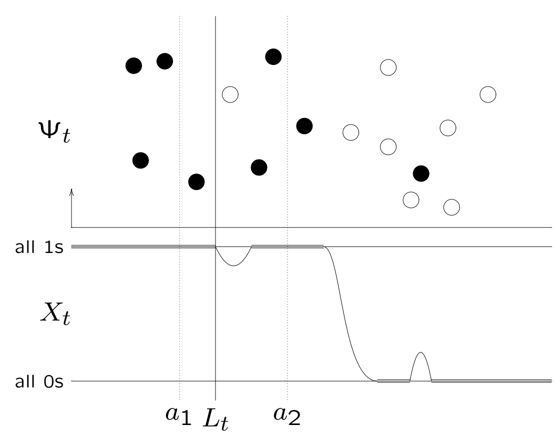
When $-\infty < L_t \leq R_t < +\infty$, it looks like:



An Interface of X_t

The processes L_t and R_t really describe X_t. For each t, we almost surely have

$$X_t((-\infty, a) \times \{0\}) = 0 \iff a \leq L_t$$
$$X_t((b, +\infty) \times \{1\}) = 0 \iff b \geq R_t$$



Path Properties of L_t and R_t

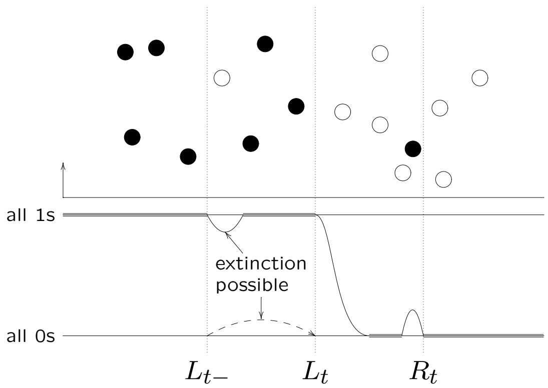
Theorem 2. *Almost surely, the processes*

$$L_t := \inf\{\widehat{Z}_u(t) : \kappa_u(t) = 0\}$$
$$R_t := \sup\{\widehat{Z}_u(t) : \kappa_u(t) = 1\}$$

are càdlàg with

$$L_{t-} \le L_t \le R_t \le R_{t-}$$

for all $t \ge 0$.



Coalescing of the Interface

Define

$$\mathfrak{T} := \inf\{t \ge 0 : L_t = R_t\}$$

an $\mathfrak{F}_t^{\Psi}$-stopping time.

Theorem 3. *Almost surely, on* $\{\mathfrak{T} < \infty\}$*, we have* $L_{\mathfrak{T}+t} = R_{\mathfrak{T}+t}$ *for all* $t \in \mathbb{R}_+$*, and the* $\tilde{E} \cup \{\Delta\}$*-valued process*

$$I_t := \begin{cases} L_{\mathfrak{T}+t} & \mathfrak{T} < \infty \\ \Delta & \mathfrak{T} = \infty \end{cases}$$

is a continuous, time-homogeneous, Borel, right process (with respect to $\mathfrak{G}_t := \mathfrak{F}^L_{\mathfrak{T}+t}$*) with law*

$$\mathbb{P}\Big[I_t \ge z \;\Big|\; I_0 = x\Big] = \mathsf{P}^z(Z_t < x)$$

Corollary 4. *If* Z *has generator* A *then* I *has generator* A_I *with*

$$\int f' A g + \int g' A_I f = 0$$

Example: Brownian Motion

- $m(dx) = dx$
- Generators:

$$\widehat{A} f(x) = \frac{1}{2} f''(x)$$
$$A_I g(x) = \frac{1}{2} g''(x)$$

- Processes:

$$Z_t = B_t$$
$$I_t = B_t$$

Example: General Diffusion

A general diffusion with generator

$$\widehat{A} f(x) = \frac{a(x)}{2} f''(x) + b(x) f'(x)$$

is usually self-dual with respect to its speed measure

$$m(dx) = m_0 e^{2\int_0^x \frac{b(y)}{a(y)} dy} a^{-1}(x) dx$$

So, by Corollary 4,

$$A_I g(x) = \frac{a(x)}{2} g''(x) + \left(\frac{a'(x)}{2} - b(x)\right) g'(x)$$

In particular, if $a(x) \equiv a > 0$, general $b(x)$ then A_I is diffusion with same diffusion rate and opposite drift.

Example: Ornstein-Uhlenbeck

- $m(dx) = e^{-x^2/2}dx$
- Generators:

$$\hat{A}f(x) = \frac{1}{2}f''(x) - \frac{1}{2}xf'(x)$$
$$A_I g(x) = \frac{1}{2}g''(x) + \frac{1}{2}xg'(x)$$

- Processes:

$$Z_t = e^{-t/2}\left(Z_0 + B\left(e^t - 1\right)\right)$$
$$I_t = e^{t/2}\left(I_0 + B\left(1 - e^{-t}\right)\right)$$

- SPDEs:

$$Z_t = Z_0 + B_t - \int_0^t \frac{1}{2}Z_s ds$$
$$I_t = I_0 + B_t + \int_0^t \frac{1}{2}I_s ds$$

Example: Stochastic Exponential

- $m(dx) = x^{-(2c+1)}dx$
- Generators:

$$\hat{A}f(x) = \frac{1}{2}x^2 f''(x) + \left(\frac{1}{2} - c\right)xf'(x)$$
$$A_I g(x) = \frac{1}{2}x^2 g''(x) + \left(\frac{1}{2} + c\right)xg'(x)$$

- Processes:

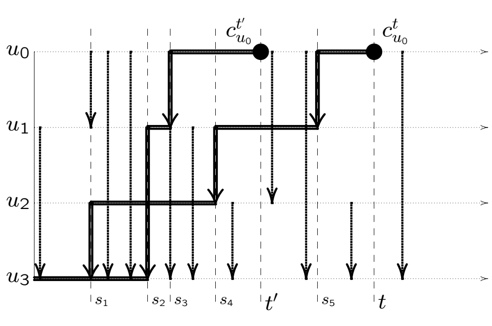
$$Z_t = Z_0 e^{B_t - ct}$$
$$I_t = I_0 e^{B_t + ct}$$

- SPDEs:

$$Z_t = Z_0 + \int_0^t Z_s dB_s + \left(\frac{1}{2} - c\right)\int_0^t Z_s ds$$
$$I_t = I_0 + \int_0^t I_s dB_s + \left(\frac{1}{2} + c\right)\int_0^t I_s ds$$

Recall: Ancestral Chains

The t-chain of u is the chain of lookdowns backward in time that determine u's type at time t.



$$c_{u_0}^t = u_0 \xrightarrow{s_5} u_1 \xrightarrow{s_4} u_2 \xrightarrow{s_1} u_3$$
$$c_{u_0}^{t'} = u_0 \xrightarrow{s_3} u_1 \xrightarrow{s_2} u_3$$

Ancestral Paths

For a (finite) chain

$$c_{u_0}^{t_0} = u_0 \xrightarrow{t_1} \dots \xrightarrow{t_m} u_m$$

we actually *define* $\kappa_{u_0}(t_0) := \kappa_{u_m}^0$.

Writing

$$\beta_{u_0}^{t_0}(t) := \begin{cases} u_k & \forall t \in [t_{k+1}, t_k),\ 0 \le k \le m-1 \\ u_m & \forall t \in [0, t_m) \end{cases}$$

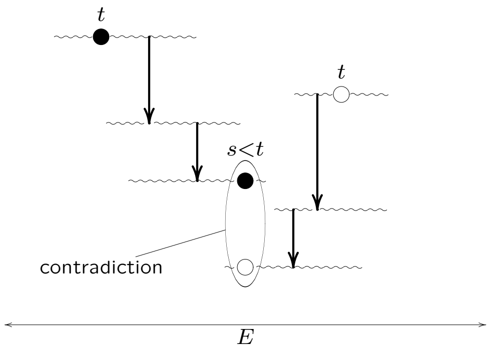
for the ancestral level process, we may define the *ancestral path* by

$$\hat{Z}_{u_0}^{\beta, t_0}(t) := \hat{Z}_{\beta_{u_0}^{t_0}(t)}(t) \quad \forall t \le t_0$$

Note that the *ancestral type* is constant:

$$\kappa_{u_0}^{\beta, t_0}(t) := \kappa_{\beta_{u_0}^{t_0}(t)}(t) \equiv \kappa_{u_m}^0 \quad \forall t \le t_0$$

Ancestral Paths of Different Types Never Meet



Lemma 5. *Almost surely, for all* $u, v \in \mathfrak{U}$ *and* $t > 0$, *if* $\kappa_u(t) \neq \kappa_v(t)$ *and* $\widehat{Z}_u(t) < \widehat{Z}_v(t)$, *then we have*

$$\widehat{Z}_u^{\beta,t}(s) < \widehat{Z}_v^{\beta,t}(s) \quad \forall s \leq t$$

Lots of Loiterers

Lemma 6. *Almost surely, for all* $(a, b) \subseteq E$ *and* $t \in \mathbb{R}_+$, *we have some* δ *such that*

$$\{u \in \mathfrak{U} : \widehat{Z}_u[t-\delta, t+\delta] \subseteq (a, b)\}$$

is infinite. In fact, δ *can be taken to be a deterministic function of* (a, b).

Corollary 7. *Almost surely, for all* $(a, b) \subseteq E$ *and* $t \in \mathbb{R}_+$, *there exists* $u \in \mathfrak{U}$ *and* $\delta > 0$ *such that*

$$\widehat{Z}_u(t-\delta, t+\delta) \subseteq (a, b)$$

and

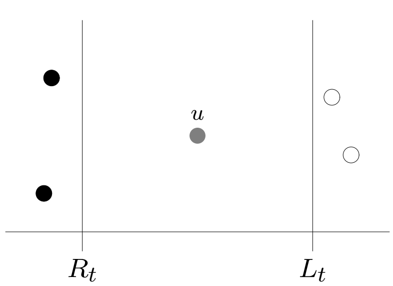
$$\lfloor u \rfloor_r = u \quad \forall r \in (t-\delta, t+\delta)$$

Moreover, if $\exists v \in \mathfrak{U}$ *with* $\widehat{Z}_v(t) \in (a, b)$ *and* $\kappa_v(t) = k$, *then* u *and* δ *may be chosen so* $\kappa_u(r) = k$ *on* $r \in (t-\delta, t+\delta)$.

Proof of Theorem 2

First, $L_t \leq R_t$ for all t.

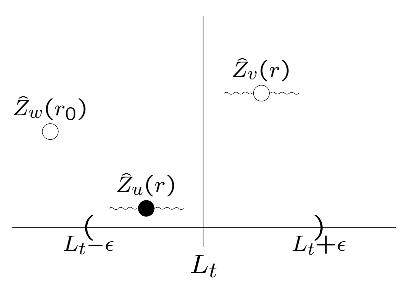
Suppose $R_t < L_t$. By Lemma 6, there are ∞-many particles in (R_t, L_t). Let u be one. What is $\kappa_u(t)$?



Proof of Theorem 2

Second, L_t is right continuous with $\overline{\lim}_{s \uparrow t} L_s \leq L_t$.

Say $L_t \in E_0$. By Corollary 7, $\forall \epsilon > 0 \, \exists \delta > 0$ such that for $r \in (t-\delta, t+\delta)$ we have:



Particle v ensures $L_r \leq L_t + \epsilon$ for all $r \in (t-\delta, t+\delta)$.

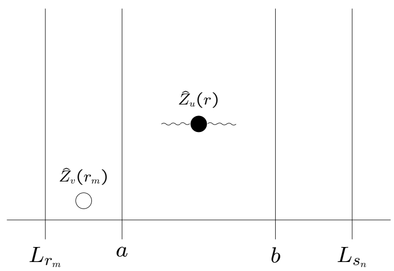
Particle u forces $L_r \geq L_t - \epsilon$ for $r \in [t, t+\delta)$. Otherwise, at some $r_0 \in [t, t+\delta)$, there'd be a w as above. Trace back ancestry of w and u to time t and invoke Lemma 5 giving the contradictory

$$\widehat{Z}_w^{\beta,r_0}(t) < \widehat{Z}_u^{\beta,r_0}(t) = \widehat{Z}_u(t) < L_t$$

Proof of Theorem 2

Third, $\underline{\lim}_{s\uparrow t} L_s = \overline{\lim}_{s\uparrow t} L_s$.

Suppose not. Then, $\exists r_m \uparrow t, s_n \uparrow t$ with $L_{r_m} < a < b < L_{s_n}$ for some $a, b \in E_0$. By Corollary 7, $\exists \delta > 0$ such that for $r \in (t-\delta, t+\delta)$ we have:

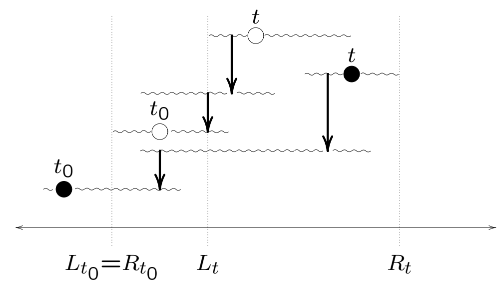


- Pick n such that $\forall n' \geq n$, $s_{n'}, r_{n'} \in (t-\delta, t+\delta)$; pick m such that $r_m > s_n$.
- Because $\widehat{Z}_u(s_n) < L_{s_n}$, we have $\kappa_u(r) \equiv 1$.
- Because $L_{r_m} < a$, there exists v as above.
- Trace ancestry of u and v from r_m back to s_n, and $\widehat{Z}_v^{\beta, r_m}(s_n) < \widehat{Z}_u^{\beta, r_m}(s_n) < L_{s_n}$, contradicting the type of particle v.

Proof of Theorem 3

First, for an almost sure set Ω', we have $\{\mathfrak{T} \leq t\} \cap \Omega' \subseteq \{L_t = R_t\} \subseteq \{\mathfrak{T} \leq t\}$.

Take Ω' the set on which ancestral paths of different types don't meet. For $t > \mathfrak{T}(\omega)$, we have $L_{t_0} = R_{t_0}$ for some $t_0 \leq t$ by the definition of $\mathfrak{T}$. Suppose $L_t < R_t$. Then, we have:



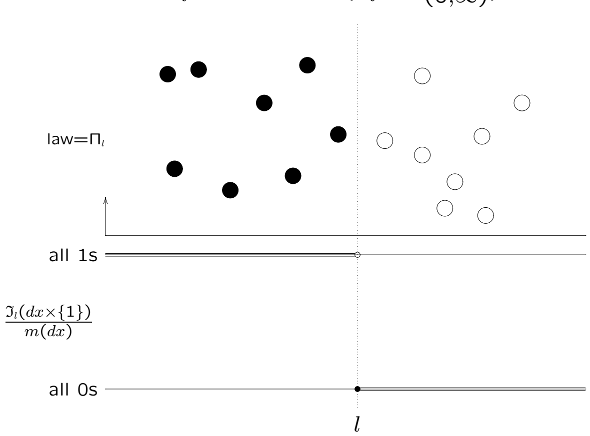
a contradiction to Ω'. So, $L_t = R_t$ for all $t > \mathfrak{T}(\omega)$ and so for $t \geq \mathfrak{T}(\omega)$ by right continuity.

Proof of Theorem 3

For $l \in \tilde{E}$, write $\mathfrak{I}_l$ for the measure on $E \times \{0, 1\}$ given by

$$\mathfrak{I}_l(A \times \{0\}) = m(A \cap [l, \sup E))$$
$$\mathfrak{I}_l(A \times \{1\}) = m(A \cap (\inf E, l))$$

and write $\Pi_l := \text{Poisson}(\mathfrak{I}_l \times \ell_{(0,\infty)})$.



Proof of Theorem 3

Here's what X_t looks like after the interface coalesces:

Lemma 8. *$\mathfrak{T}$ is an $\mathfrak{F}_t$-stopping time with*

$$\mathbb{P}(X_t \neq \mathfrak{I}_{L_t}; \mathfrak{T} \leq t) = 0$$

Note $L_t = h(\Psi_t)$ for some measurable h. Define the semigroup

$$P_t^I f(l) := \int \mathbb{P}_t f \circ h(\psi) \Pi_l(\psi) = \mathbb{P}^{\Pi_l} f(L_t)$$

Proof of Theorem 3

Second, an "approximation" of the simple Markov property of I_t.

Lemma 9. *For all $s \le t$ and $A \in \mathfrak{F}_s$,*

$$\mathsf{E}[f(L_t); \mathfrak{T} \le s, A] = \mathsf{E}[P^I_{t-s} f(L_s); \mathfrak{T} \le s, A]$$

Proof.

$$\begin{aligned}
\mathsf{E}[f(L_t); \mathfrak{T} \le s | \mathfrak{F}_s] &= \mathsf{E}[\mathsf{E}[f \circ h(\Psi_t) | \mathfrak{F}^\Psi_s] 1_{\mathfrak{T} \le s} | \mathfrak{F}_s] \\
&= \mathsf{E}[1_{\mathfrak{T} \le s} \mathbb{P}_{t-s} f \circ h(\Psi_s) | \mathfrak{F}_s] \\
&= 1_{\mathfrak{T} \le s} \int \mathbb{P}_{t-s} f \circ h(\psi) \Pi_{L_s}(d\psi) \\
&= 1_{\mathfrak{T} \le s} P^I_{t-s} f(L_s)
\end{aligned}$$

□

Approximating $\mathfrak{T}$ from above by random times $\mathfrak{T}_n$ each with countable range, we can use the continuity of $P^I_t f$ over a rich class of functions $f \in \mathfrak{K}$ to show

$$\mathsf{E}[f(L_{\mathfrak{T}+t}); \mathfrak{T} < \infty | \mathfrak{F}^L_{\mathfrak{T}+s}] = 1_{\mathfrak{T} < \infty} P^I_{t-s} f(L_{\mathfrak{T}+s})$$

and use a monotone class theorem to extend to $f \in b\mathcal{E}$.

Proof of Theorem 3

Third, the law of I_t.

Note $\mathbb{P}^{\Pi_l}(\mathfrak{T} = 0) = 1$, so

$$\begin{aligned}
\int_E e^{-\alpha z} \mathsf{P}^l(I_t < z) m(dz) \\
&= \int_E e^{-\alpha z} \mathbb{P}^{\Pi_l}(L_t < z) m(dz) \\
&= \mathbb{P}^{\Pi_l} \int_{L_t}^{\sup E} e^{-\alpha z} m(dz) \\
&= \mathbb{P}^{\Pi_l} \int_E e^{-\alpha z} X_t(dz \times \{0\}) \\
&= \int_E e^{-\alpha z} \mathsf{P}^z(Z(t) \ge l) m(dz)
\end{aligned}$$

with the last equality following from a duality result of (DEFKZ, 2000).

Inverting the Laplace transform gives the law.