Interface Solutions of Lookdown Particle Systems, Part II: Limiting Measures and SPDEs

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General Lookdown Systems

Definition 1. Given a countable index set $\mathfrak{U}$, a *genealogy* is a timed, directed graph

$$\mathfrak{G} \subseteq (0, \infty) \times \mathfrak{U} \times \mathfrak{U}$$

For $(t, u, v) \in \mathfrak{G}$, we write $u \xrightarrow[\mathfrak{G}]{t} v$.

Interpretation: Particles indexed by $\mathfrak{U}$ have initial types $\kappa_u^0 \in \{0, 1\}$. At time t, particle u looks (down) at particle v and copies its type just prior to time t.

If $\mathfrak{G}$ satisfies some axioms, then there exist jump processes (càdlàg in the discrete topology) started at κ_u^0 with

$${}^{\mathfrak{G}}\kappa_u(t) = \begin{cases} {}^{\mathfrak{G}}\kappa_v(t-), & \text{if } u \xrightarrow{t} v \\ {}^{\mathfrak{G}}\kappa_u(t-), & \text{otherwise} \end{cases}$$

Particle Motion with Dual

Let Z, $\widehat{Z}$ be continuous Markov on interval $E \subseteq \mathbb{R}$ with semigroups P_t and $\widehat{P}_t$ satisfying $P_t 1 = \widehat{P}_t 1 = 1$ and the duality

$$\int f P_t g \, dm = \int g \widehat{P}_t f \, dm$$

for all $f, g \in p\mathcal{E}$ and some diffuse Borel m with $0 < m(a, b) < \infty$ for all $a < b$, $a, b \in E$.

Hypothesis 2. There exists $E_n \to E$ relatively open in E with $m(E_n) < \infty$ and

$$\int_E \widehat{\mathsf{P}}^z \{\sigma_{E_n} \leq t\} m(dz) < \infty$$

where

$$\sigma_A := \inf\{t \geq 0 : \widehat{Z}(t) \in A\}$$

Note: The notation and framework borrow heavily from (DEFKZ, 2000).

An Interacting Particle System

Let ν_0 be a measure on $E \times \{0, 1\}$ with $\nu_0(\cdot \times \{0, 1\}) = m$, and take

$$\Psi_0 := \sum_{u \in \mathfrak{U}} \delta_{(\widehat{Z}_u^0, \kappa_u^0, u)} \sim \text{Poisson}\left(\nu_0 \times \ell_{(0,\infty)}\right)$$

Then, let $\widehat{Z}_u$ given Ψ_0 be conditionally independent copies of $\widehat{Z}$ each started at $\widehat{Z}_u^0$.

Finally, for a genealogy $\mathfrak{G}$ satisfying the axioms, define the system

$${}^{\mathfrak{G}}\Psi_t := \sum_{u \in \mathfrak{U}} \delta_{(\widehat{Z}_u(t), {}^{\mathfrak{G}}\kappa_u(t), u)}$$

Particle Systems with Slow Local Interactions (B., 2002)

Let L_{uv} be a continuous additive functional of $(\hat{Z}_u, \hat{Z}_v)$ such that

$$L_{uv}(s) - L_{uv}(r) > 0 \iff \exists t \in (r,s)\,[\hat{Z}_u(t) = \hat{Z}_v(t)]$$

That is, L_{uv} is a continuous, monotone increasing processes that increases when (and only when) $\hat{Z}_u(t) = \hat{Z}_v(t)$.

For $\lambda \in (0,\infty)$, define

$$\mathfrak{G}^\lambda := \{(t,u,v) : \Delta V_{uv}(t) > 0, u > v\}$$

where $V_{uv}(t) := N_{uv}(\lambda L_{uv}(t))$.

Particle Systems with Fast Local Interactions (DEFKZ, 2000)

Let

$$[u]_t := \{v \in \mathfrak{U} : \hat{Z}_u(t) = \hat{Z}_v(t)\}$$

It turns out $[u]_t$ always has a minimum. Denote it by $\lfloor u \rfloor_t$.

Define

$$\mathfrak{G}^\infty = \{(t,u,\lfloor u \rfloor_t) : \lfloor u \rfloor_t \neq u,\, u \in \mathfrak{U}, t > 0\}$$

If type processes exists, they satisfy

$$\hat{Z}_u(t) = \hat{Z}_v(t) \quad \Longrightarrow \quad \kappa_u(t) = \kappa_v(t)$$

so, in spirit, this is really the genealogy

$$\{(t,u,v) : \hat{Z}_u(t) = \hat{Z}_v(t),\, u > v\}$$

Existence of $\mathfrak{G}^\lambda$ Systems, $\lambda \leq \infty$

Theorem 3. *The genealogies $\mathfrak{G}^\lambda$, $\lambda \leq \infty$ satisfy the axioms. That is, there exist càdlàg type processes ${}^\lambda\kappa_u$ for these genealogies and càdlàg particle systems*

$${}^\lambda\Psi_t := \sum_{u \in \mathfrak{U}} \delta_{(\hat{Z}_u(t), {}^\lambda\kappa_u(t), u)}$$

J_1 Convergence

Theorem 4. *For $n \in \mathbb{N}$, $M > 0$, there exists a family $(\delta_\lambda)_{\lambda \in (0,\infty)}$ of time changes such that for all $T > 0$,*

$$\lim_{\lambda\to\infty} \sup_{t \leq T} |\delta_\lambda(t) - t| = 0$$

and

$$\lim_{\lambda\to\infty} \sup_{t \leq T} \sup_{u \in S_{M,n}} |{}^\lambda\kappa_u(t) - {}^\infty\kappa_u(\delta_\lambda(t))| = 0$$

where

$$S_{M,n} := \{u \in \mathfrak{U} \cap (0,M) : \hat{Z}_u(0) \in E_n\}$$

Exchangeability of Levels

Fix $\lambda \le \infty$ (and drop it from the notation). For $M > 0$, define

$$\mathfrak{F}_t^M := \sigma\Big\{ \sum_{u\in\mathfrak{U}\cap[M,\infty)} \delta_{(\hat{Z}_u(r),\kappa_u(r),u)}, \\ \sum_{u\in\mathfrak{U}\cap(0,M)} \delta_{(\hat{Z}_u(r),\kappa_u(r))} \quad : r \le t \Big\}$$

Lemma 5. *We have*

$$\mathfrak{L}\Big[\sum_{u\in\mathfrak{U}\cap(0,M)} \delta_{(\hat{Z}_u(t),\kappa_u(t),u)} \,\Big|\, \mathfrak{F}_t^M\Big] \\ = \mathfrak{L}\Big[\sum_{u\in\mathfrak{U}\cap(0,M)} \delta_{(\hat{Z}_u(t),\kappa_u(t),\Upsilon_u)} \,\Big|\, \mathfrak{F}_t^M\Big]$$

for Υ_u conditionally iid uniform on $(0, M)$ given $\mathfrak{F}_t^\Psi$.

Convergence of X^M

Note: $\mathfrak{F}_t^M \downarrow$ in M, so take $\mathfrak{F}_t^\infty := \bigcap_{M>0} \mathfrak{F}_t^M$.

Define

$$X_t^M := \frac{1}{M} \sum_{u\in\mathfrak{U}\cap(0,M)} \delta_{(\hat{Z}_u(t),\kappa_u(t))}$$

Theorem 6. *For $h \in B(E \times \{0,1\})$ with support in $E_n \times \{0,1\}$,*

$$X_t^M(h) \underset{M\to\infty}{\longrightarrow} \mathsf{E}[X_t^1(h)|\mathfrak{F}_t^\infty] =: X_t(h)$$

almost surely and in L_p, $p \ge 1$.

Proof of Theorem 6

By Lemma 5,

$$\begin{aligned} &\mathsf{E}[X_t^1(h)|\mathfrak{F}_t^M] \\ &= \mathsf{E}\Big[\sum_{u\in\mathfrak{U}\cap(0,M)} 1_{u<1} h(\hat{Z}_u(t),\kappa_u(t)) \,\Big|\, \mathfrak{F}_t^M\Big] \\ &= \mathsf{E}\Big[\sum_{u\in\mathfrak{U}\cap(0,M)} 1_{\Upsilon_u<1} h(\hat{Z}_u(t),\kappa_u(t)) \,\Big|\, \mathfrak{F}_t^M\Big] \\ &= \frac{1}{M} \sum_{u\in\mathfrak{U}\cap(0,M)} h(\hat{Z}_u(t),\kappa_u(t)) \\ &= X_t^M(h) \end{aligned}$$

But, $\mathsf{E}[X_t^1(h)|\mathfrak{F}_t^M]$ is a backwards martingale in $M \in (0,\infty]$, so

$$X_t^M(h) = \mathsf{E}[X_t^1(h)|\mathfrak{F}_t^M] \to \mathsf{E}[X_t^1(h)|\mathfrak{F}_t^\infty]$$

Conditional Poisson Structure

Theorem 7.

$$\mathfrak{L}[\Psi_t \mid \mathfrak{F}_t^\infty] = \text{Poisson}(X_t \times \ell_{(0,\infty)})$$

Proof. Let f be supported on $E_n \times \{0,1\} \times (0, M_0)$. For $M \ge M_0$, Lemma 5 and Taylor give

$$\begin{aligned} &\mathsf{E}[e^{\theta\Psi_t(f)}|\mathfrak{F}_t^M] \\ &= \mathsf{E}\Big[\exp\Big(\theta \sum_{u\in\mathfrak{U}\cap(0,M)} f(\hat{Z}_u(t),\kappa_u(t),\Upsilon_u)\Big) \,\Big|\, \mathfrak{F}_t^M\Big] \\ &= \prod_{u\in\mathfrak{U}\cap(0,M)} \frac{1}{M}\int_0^M \exp(\theta f(\hat{Z}_u(t),\kappa_u(t),v))\,dv \\ &= \exp\Big(\sum_{u\in\mathfrak{U}\cap(0,M)} \log\Big(1+ \\ &\qquad \frac{1}{M}\int_0^\infty \Big(e^{\theta f(\hat{Z}_u(t),\kappa_u(t),v)} - 1\Big)dv\Big)\Big) \\ &= \exp\Big(X_t^M\Big(\int_0^\infty \Big(e^{\theta f(\cdot,\cdot,v)} - 1\Big)dv + O\Big(\frac{1}{M}\Big)\Big)\Big) \\ &\to \exp\Big(X_t\Big(\int_0^\infty \Big(e^{\theta f(\cdot,\cdot,v)} - 1\Big)dv\Big)\Big) \end{aligned}$$

□

An Itô Identity

Say that

$$M_u^f(t) := f(\widehat{Z}_u(t)) - f(\widehat{Z}_u(0)) - \int_0^t \widehat{A} f(\widehat{Z}_u(s))ds$$

is a martingale. Then, we have the identity

$$X_t^M(fg) - X_0^M(fg) - \int_0^t X_s^M((\widehat{A}f)g)ds$$
$$= \frac{1}{M} \sum_{u \in \mathfrak{U} \cap (0,M)} \int_0^t g(\kappa_u(s-))dM_u^f(s)$$
$$+ \frac{1}{M} \sum_{\substack{u,v \in \mathfrak{U} \cap (0,M) \\ u>v}} \int_0^t \big(f(\widehat{Z}_v(s-))g(\kappa_v(s-)) - f(\widehat{Z}_u(s-))g(\kappa_u(s-))\big)dV_{uv}(s)$$

(with V_{uv} appropriately defined for $\lambda = \infty$).

Claim 8. This last term is an $\mathfrak{F}_t^M$-martingale.

Martingale Problem

For $M = 1$, the identity gives

$$X_t^1(fg) - X_0^1(fg) - \int_0^t X_s^1((\widehat{A}f)g)ds$$

an $\mathfrak{F}_t^1$-martingale. If $X_t := \mathsf{E}[X_t^1(\cdot)|\mathfrak{F}_t^\infty]$ exists as a "nice" process, a well-known result gives

$$X_t(fg) - X_0(fg) - \int_0^t X_s((\widehat{A}f)g)ds$$

an $\mathfrak{F}_t^\infty$-martingale.

Formally, if $u(x,t)m(dx) = X_t(dx \times \{1\})$, then

$$\dot{u} = Au + \dot{M}$$

where $\dot{M}$ is some martingale noise.

Brownian Case (B., 2002)

In the case

- $\lambda < \infty$;
- $\widehat{Z}$, Z standard Brownian;
- m Lebesgue on $E = \mathbb{R}$;
- L_{uv} martingale local time at 0 of $\widehat{Z}_u - \widehat{Z}_v$.

then

$$\dot{u} = \frac{1}{2}\Delta u + \sqrt{4\lambda u(1-u)}\dot{W}$$

for $\dot{W}$ a space-time white noise.

Space-Time Scaling (Tribe, 1995)

The SPDE given by

$$\dot{u} = \frac{1}{2}\Delta u + \sqrt{u(1-u)}\dot{W}$$

has a compact support property. Define

$$L_t := \inf\{x : u(x,t) < 1\}$$
$$R_t := \sup\{x : u(x,t) > 0\}$$

Then, if $-\infty < L_t \leq R_t < \infty$ at $t = 0$ then it holds for all $t \geq 0$.

Take a space-time scaling

$$v_t^{(n)}(x) := u_{n^2 t}(nx)$$

In the limit $n \to \infty$, $R_t^{(n)} - L_t^{(n)} \xrightarrow{p} 0$ and $R_t^{(n)} \xrightarrow{d} B_t$.

Another View of the Scaling

If $u_t(x)$ solves

$$\dot{u} = \frac{1}{2}\Delta u + \sqrt{u(1-u)}\dot{W}$$

then $v_t^{(n)}(x) = u_{n^2 t}(nx)$ solves

$$\dot{v}^{(n)} = \frac{1}{2}\Delta v^{(n)} + \sqrt{nv^{(n)}\left(1 - v^{(n)}\right)}\dot{W}$$

Thus, Tribe's result is really:

Theorem 9. *Define*

$$\begin{aligned} {}^\lambda L_t &:= \inf\{x : {}^\lambda X_t((-\infty, x) \times \{0\}) > 0\} \\ {}^\lambda R_t &:= \sup\{x : {}^\lambda X_t((x, \infty) \times \{1\}) > 0\} \end{aligned}$$

Then, as $\lambda \to \infty$ in the Brownian case, both ${}^\lambda L_t$ and ${}^\lambda R_t$ converge to the same Brownian motion.

This begs the question, why not look at ${}^\infty L_t$ and ${}^\infty R_t$ directly?

But, Be Careful!

We've shown

- For each box $E_n \times \{0,1\} \times (0, M)$, there is a family of time changes δ_λ. On each compact time interval as $\lambda \to \infty$, we have $\delta_\lambda \to id$ uniformly and eventually ${}^\lambda\Psi_t = {}^\infty\Psi_{\delta_\lambda(t)}$ on the box (and so ${}^\lambda X_t^M = {}^\infty X_{\delta_\lambda(t)}^M$ on E_n);
- For fixed λ, ${}^\lambda X_t^M(A) \to {}^\lambda X_t(A)$ as $M \to \infty$ for all compact A.

but we can't conclude ${}^\lambda X \to {}^\infty X$ in a sense that would give

$$\begin{aligned} {}^\lambda L &\to {}^\infty L \\ {}^\lambda R &\to {}^\infty R \end{aligned}$$

Interface Solutions of Lookdown Particle Systems, Part III: Interfaces of Fast-Interaction Models

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