Lookdown Particle Systems with Local Interactions

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General Lookdown Systems

Definition 1. Given a countable index set $\mathfrak{U}$, a *genealogy* is a timed, directed graph

$$\mathfrak{G} \subseteq (0,\infty) \times \mathfrak{U} \times \mathfrak{U}$$

For $(t,u,v) \in \mathfrak{G}$, we write $u \xrightarrow[\mathfrak{G}]{t} v$.

Interpretation: Particles indexed by $\mathfrak{U}$ have initial types $\kappa_u^0 \in \mathbb{K}$. At time t, particle u looks (down) at particle v and copies its type just prior to time t.

Biological Interpretation: In population with fixed size, particle v gives birth at time t to an offspring (with the same type) that replaces particle u in the population.

Note: Often $\mathfrak{U} \subseteq (0,\infty)$ and $u \xrightarrow{t} v$ only if $u > v$, justifying the term look*down*.

Moran Model

Simple n-particle Moran model:

$$\mathfrak{U} = \{1,\ldots,n\}$$
$$\mathfrak{G} = \{(t,u,v) : \Delta N_{uv}(t) = 1,\ u \neq v\}$$

Countable construction of Fleming-Viot process (Donnelly & Kurtz, 1996):

$$\mathfrak{U} = \mathbb{N}$$
$$\mathfrak{G} = \{(t,u,v) : \Delta N_{uv}(2t) = 1,\ u > v\}$$

Axioms $(A0)$–$(A3)$

$(A0)$ $\quad u \not\rightarrow u$

$(A1)$ $\quad u \xrightarrow{t} v_1,\ u \xrightarrow{t} v_2 \implies v_1 = v_2$

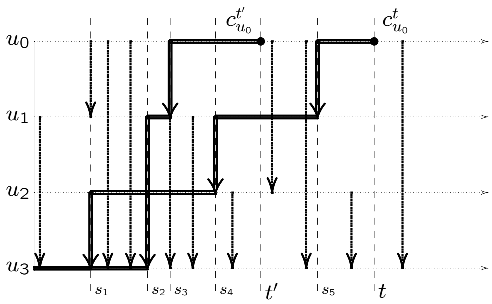
$(A2\uparrow)$ $\quad u \xrightarrow{t_n} v_n,\ t_n \uparrow t \implies \exists v\,[u \xrightarrow{t} v]$

$(A3\uparrow)$ $\quad u \xrightarrow{t} v \implies (\exists s < t)[v \overset{(s,t)}{\not\rightarrow}]$

This isn't enough to guarantee $\mathfrak{G}$ is meaningful, but it allows us to construct. . .

Ancestral Chains

Definition 2. The t-chain of u (written ${}^{\mathfrak{G}}c_u^t$) indicates the ancestral path backward in time of particle u starting at t.



$$c_{u_0}^t = u_0 \xrightarrow{s_5} u_1 \xrightarrow{s_4} u_2 \xrightarrow{s_1} u_3$$
$$c_{u_0}^{t'} = u_0 \xrightarrow{s_3} u_1 \xrightarrow{s_2} u_3$$

Defining Types

If a chain exists and is finite

$$c_{u_0}^{t_0} = u_0 \xrightarrow{t_1} \dots \xrightarrow{t_m} u_m$$

we write $\bar{c}_{u_0}^{t_0} := u_m$ and *define* $\kappa_{u_0}(t_0) := \kappa^0_{\bar{c}_{u_0}^{t_0}}$.
We also write

$$\beta_{u_0}^{t_0}(t) := \begin{cases} u_k & \forall t \in [t_{k+1}, t_k),\ 0 \le k \le m-1 \\ u_m & \forall t \in [0, t_m) \end{cases}$$

Axioms (A) Imply Chains Exist

Lemma 3. *Axioms* $(A0)$, $(A1)$, $(A2\uparrow)$, *and* $(A3\uparrow)$ *imply for every* $u_0 \in \mathfrak{U}$ *and* $t_0 > 0$ *there exists a (possibly infinite) unique chain*

$$c_{u_0}^{t_0} = u_0 \xrightarrow{t_1} u_1 \xrightarrow{t_2} \dots$$

with $t_0 \ge t_1 > t_2 > \dots > 0$ *and* $u_{k-1} \ne u_k$.

Proof. Write $\tau_u = \{t : u \xrightarrow{t}\}$. If $\tau_{u_0} \cap (0, t_0]$ is empty, final chain is $c_{u_0}^{t_0} = u_0$. Otherwise, let t_1 be its maximum (by $(A2\uparrow)$), and chain-so-far is

$$u \xrightarrow{t_1} u_1$$

Once we have

$$u_0 \xrightarrow{t_1} \dots \xrightarrow{t_m} u_m$$

for $t_0 \ge t_1 > \dots > t_m > 0$, if $\tau_{u_m} \cap (0, t_m)$ is empty, that's final chain. Otherwise, let t_{m+1} be its maximum (by $(A2\uparrow)$, $(A3\uparrow)$), and extend chain with $u_m \xrightarrow{t_{m+1}} u_{m+1}$. $\square$

Axioms (B) and (C)

(B) All chains are finite.

(C) For $t_n \to t$ monotone, $\bar{c}_u^{t_n}$ is eventually constant (and equal to $\bar{c}_u^t$ if $t_n \downarrow$).

Obviously, (B) implies type processes exist while (C) implies they are càdlàg (in the discrete topology and so càdlàg with well separated jumps).

The Specifics: Motion and Dual

Let Z, $\widehat{Z}$ be continuous Markov on internal $E \subseteq \mathbb{R}$ with semigroups P_t and $\widehat{P}_t$ satisfying $P_t 1 = \widehat{P}_t 1 = 1$ and the duality

$$\int f P_t g \, dm = \int g \widehat{P}_t f \, dm$$

for all $f, g \in p\mathcal{E}$ and some diffuse Borel m with $0 < m(a,b) < \infty$ for all $a < b$, $a, b \in E$.

Hypothesis 4. There exists $E_n \to E$ relatively open in E with $m(E_n) < \infty$ and

$$\int_E \widehat{\mathsf{P}}^z \{\sigma_{E_n} \le t\} m(dz) < \infty$$

where

$$\sigma_A := \inf\{t \ge 0 : \widehat{Z}(t) \in A\}$$

Note: The notation and framework borrow heavily from (DEFKZ, 2000).

The Specifics: A Particle System

Let ν_0 be a measure on $E \times \mathbb{K}$ with $\nu_0(\cdot \times \mathbb{K}) = m$, and take

$$\Psi_0 := \sum_{u \in \mathfrak{U}} \delta_{(\widehat{Z}_u^0, \kappa_u^0, u)} \sim \text{Poisson}\left(\nu_0 \times \ell_{(0,\infty)}\right)$$

Finally, let $\widehat{Z}_u$ given Ψ_0 be conditionally independent copies of $\widehat{Z}$ each started at $\widehat{Z}_u^0$.

Lemma 5. *For $t > 0$, $M > 0$, $n \in \mathbb{N}$, we have*

$$\{v \in \mathfrak{U} \cap (0, M) : \widehat{Z}_v[0,t] \cap E_n \neq \emptyset\}$$

almost surely finite.

Particle Systems with Slow Local Interactions (B, 2002)

For a fixed, finite α-potential on $E \times E$, let L_{uv} be the continuous additive functional of $(\widehat{Z}_u, \widehat{Z}_v)$ having that potential. We assume that for $r < s$, we have

$$L_{uv}(s) - L_{uv}(r) > 0 \iff \exists t \in (r,s)\, [\widehat{Z}_u(t) = \widehat{Z}_v(t)]$$

That is, L_{uv} is a continuous, monotone increasing processes that increases when (and only when) $\widehat{Z}_u(t) = \widehat{Z}_v(t)$. Moreover, by Lemma 5, $L_u := \sum_{v<u} L_{uv}$ is continuous.

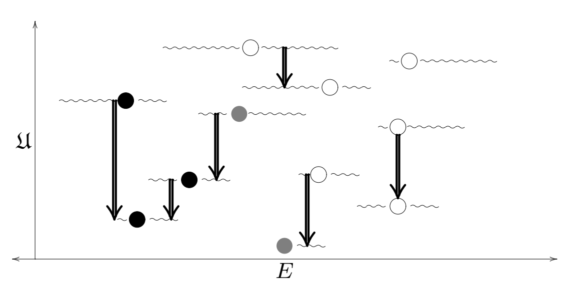
Define

$$\mathfrak{G}^{(\lambda)} := \{(t,u,v) : \Delta V_{uv}(t) > 0, u > v\}$$

where $V_{uv}(t) := N_{uv}(\lambda L_{uv}(t))$.

Note: As L_u is continuous, $(A1)$ is immediate. In fact, $|\tau_u \cap (0,t)| < \infty$, so $(A1)$, $(A2\uparrow)$, and $(A3\uparrow)$ all hold.

A Picture



Particle Systems with Fast Local Interactions (DEFKZ, 2000)

Define

$$\mathfrak{G}^{(\infty)} := \{(t,u,v) : \widehat{Z}_u(t) = \widehat{Z}_v(t),\ u > v,\ \widehat{Z}_u(t) = \widehat{Z}_w(t) \Rightarrow w \geq v\}$$

Lemma 5 implies:

- we don't miss a lookdown;
- if $t_n \uparrow t$, $u \xrightarrow{t_n} v_n$, then $\{v_n\}$ finite implies $\exists w\, [u \xrightarrow{t_n} w\, (i.o.\ n)]$ and by continuity $\widehat{Z}_u(t) = \widehat{Z}_w(t)$ implying $u \xrightarrow{t} \exists v$ (i.e., $(A2\uparrow)$ holds);
- if $u \xrightarrow{t} v$, then $v \overset{t}{\not\longrightarrow}$, and by continuity $v \overset{(r,s)}{\not\longrightarrow}$ for $r < t < s$ (i.e., $(A3\uparrow)$ holds).

All Chains Are Finite

Lemma 6. *Almost surely, all chains ${}^{\mathfrak{G}}c_u^{t_0}$ are finite (for $\mathfrak{G} = \mathfrak{G}^{(\lambda)}$ or $\mathfrak{G}^{(\infty)}$).*

Proof. ($\mathfrak{G}^{(\lambda)}$ case) Fix t_0 and define the backwards filtration

$$\widehat{\mathfrak{F}}_r := \sigma\{\mathfrak{U}, \hat{Z}_u(s), V_{uv}(s-), V_{uv}(s) : u > v \in \mathfrak{U}, s \in [t_0 - r, t_0]\}$$

For any $\widehat{\mathfrak{F}}_0$-measurable $u_0 \in \mathfrak{U}$, write its (finite or infinite) chain

$$c_{u_0}^{t_0} = u_0 \xrightarrow{t_1} u_1 \xrightarrow{t_2} \dots$$

Note $\tau_k = t_0 - t_k$ are (strictly increasing) $\widehat{\mathfrak{F}}_r$-stopping times:

$$\tau_1 = \inf\left\{s : \textstyle\sum_{v \in \mathfrak{U} \cap (0,u_0)} \Delta V_{u_0,v}(t_0 - s) = 1\right\}$$

is a stopping time, so u_1 is $\widehat{\mathfrak{F}}_{\tau_1}$-measurable. Thus,

$$\tau_2 = \inf\left\{s > \tau_1 : \textstyle\sum_{v \in \mathfrak{U} \cap (0,u_1)} \Delta V_{u_1,v}(t_0 - s) = 1\right\}$$

is a stopping time, so u_2 is $\widehat{\mathfrak{F}}_{\tau_2}$-measurable, and so on.

All Chains Are Finite

Proof. (cont.) By continuity of $\hat{Z}_u$, strong Markov property, and duality relation,

$$Z(r) := \hat{Z}_{\beta_{u_0}^{t_0}(t_0 - r)}(t_0 - r)$$

defined on $(0, t_0 \wedge \lim \tau_k)$ is a continuous Markov copy of Z. In particular, its path is contained in some E_n implying, by Lemma 5, that $\tau_k = \infty$ for some k and the chain is finite.

Now, on an almost sure set where c_u^q finite for all $u \in \mathfrak{U}$ and $q \in \mathbb{Q}_+$, for arbitrary $t \in (0,\infty)$, either $u \xrightarrow{t} u'$ implying $u' \not\longrightarrow$ on $(t-\epsilon, t)$ by $(A3\uparrow)$, or $u \overset{t}{\not\longrightarrow}$ implying $u \not\longrightarrow$ on $(t-\epsilon,t)$ by $(A2\uparrow)$. Take $q \in \mathbb{Q}_+ \cap (t - \epsilon.t)$, and $c_u^t = u \xrightarrow{t} c_{u'}^q$ or $c_u^t = c_u^q$ respectively. $\square$

Path Regularity

For $\mathfrak{G}^{(\lambda)}$, as $\tau_u \cap (0,t)$ finite, c_u^t (and so $\bar{c}_u^t$) changes at well separated time points. Therefore, $\kappa_u(t) := \kappa^0_{\bar{c}_u^t}$ are càdlàg jump processes with well separated jumps.

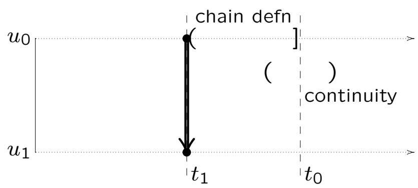
For $\mathfrak{G}^{(\infty)}$, it turns out that $\bar{c}_u^t$ (but not necessarily c_u^t) changes at well separated time points too, so $\kappa_u(t)$ are càdlàg jump processes with well separated jumps.

Path Regularity of $\mathfrak{G}^{(\infty)}$

Say we have

$$c_{u_0}^{t_0} = u_0 \xrightarrow{t_1} \dots \xrightarrow{t_m} u_m$$

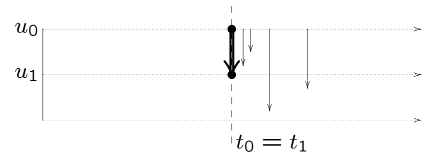
- **Case 1:** $t_0 > t_1$



Therefore, $c_{u_0}^t = c_{u_0}^{t_0}$ is constant in a neighborhood of t_0.

Path Regularity of $\mathfrak{G}^{(\infty)}$

- **Case 2:** $t_0 = t_1$, right regularity



Consider

$$S = \{w \le u_0 : \hat{Z}_w(t_0) = \hat{Z}_{u_0}(t_0)\}$$

By continuity and Lemma 5, $\exists (r,s) \ni t_0$ such that no particle in S contacts a particle in $(\mathfrak{U} \cap (0, u_0)) \setminus S$.

Then, for any $t \in (t_0, s)$, we have the finite chain

$$c_{u_0}^t = u_0 \xrightarrow{s_1} w_1 \longrightarrow \dots \xrightarrow{s_k} w_k \xrightarrow{t_0} u_1 \longrightarrow \dots$$

for $\{w_i\} \subseteq S$. So, $\bar{c}_{u_0}^t = \bar{c}_{u_1}^{t_0} = \bar{c}_{u_0}^{t_0}$.

No Simultaneous Jumps

Lemma 7. *Almost surely, $\Delta \bar{c}_u^t \neq 0$ and $\Delta \bar{c}_v^t \neq 0$ implies $u = v$.*

It is a consequence of:

Lemma 8 (Lemma 2.1 of DEFKZ). *If Y is a copy of $\hat{Z}$ started at $q \ll m$ and (T, V) is a $[0, \infty) \times E$-valued r.v. independent of Y, then*

$$P(Y(T) = V) = 0$$

Proof of Lemma 7. Roughly, if T is a jump time of $\bar{c}_w^t$, and $V := \hat{Z}_v(\bar{c}_w^t)$ is the location of any particle, then for all $u > v, w$, $Y := \hat{Z}_u$ is independent of (T, V).

As a consequence, a particle u will never hit any other particle when a lower-level $\bar{c}_w^t$ jumps. In particular, $\bar{c}_u^t$ can't jump. $\square$

Previous Constructions

- For Z_u, $\hat{Z}_u$ Brownian with L_{uv} the martingale local time at 0 of $\hat{Z}_u - \hat{Z}_v$, the $\mathfrak{G}^{(\lambda)}$-process was constructed in (B., 2002);
- For general Z_u, $\hat{Z}_u$ (not necessarily continuous), the analogue of the $\mathfrak{G}^{(\infty)}$-process was constructed in (DEFKZ, 2000), but only almost surely at each fixed t.

J_1 Convergence

Write

$$\Psi^{(\lambda)}(t) := \sum_{u \in \mathfrak{U}} \delta_{(\widehat{Z}_u(t), {}^{\lambda}\kappa_u(t), u)}$$
$$\Psi^{(\infty)}(t) := \sum_{u \in \mathfrak{U}} \delta_{(\widehat{Z}_u(t), {}^{\infty}\kappa_u(t), u)}$$

Theorem 9. *Almost surely, for all $n \in \mathbb{N}$ and $M > 0$,*

$$\Psi^{(\lambda)}\Big|_{E_n \times \mathbb{K} \times (0,M)} \underset{\lambda \to \infty}{\longrightarrow} \Psi^{(\infty)}\Big|_{E_n \times \mathbb{K} \times (0,M)}$$

in the J_1 topology on $D_{E \times \mathbb{K} \times (0,\infty)}$, particle by particle.