Interfaces of Two-Type
Continuum-Sites
Stepping-Stone Models

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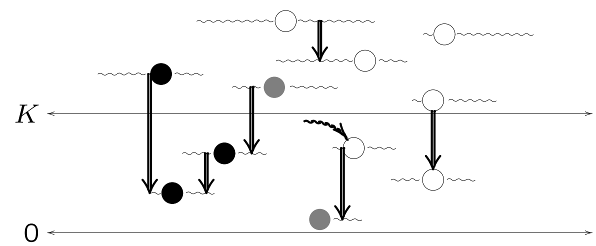
October 23, 2002

Lessons From Last Week #1



The infinite-density, ordered model embeds finite-density, symmetric models of all densities.

$$\Psi_t = \sum_{u \in \mathfrak{U}} \delta_{(\widehat{Z}_u(t), \kappa_u(t), u)}$$

If

$$X_t^K = \sum_{u \in \mathfrak{U},\, u < K} \delta_{(\widehat{Z}_u(t), \kappa_u(t))}$$

then

$$X^K =^d K\text{-density symmetric model}$$

Lessons From Last Week #2

The limiting location type measure

$$X_t = \lim_{K\to\infty} \frac{1}{K} X_t^K = \lim_{K\to\infty} \frac{1}{K} \sum_{u \in \mathfrak{U},\, u<K} \delta_{(\widehat{Z}_u(t), \kappa_u(t))}$$

- "exists" as an L_p and a.s. limit;
- is the weak limit of the (mass-scaled) K-density, symmetric model as $K \to \infty$;
- satisfies

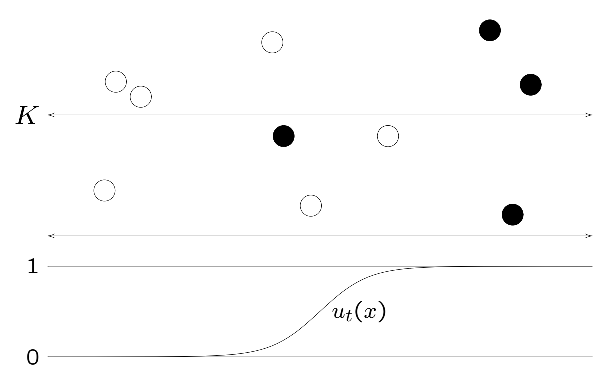
$$\mathfrak{L}\Big(\Psi_t \,\Big|\, \mathfrak{F}_t^X\Big) = \text{Poisson}(X_t \times \ell_{\mathbb{R}_+})$$

The "essence" of the infinite-density, continuum-sites, stepping-stone model is X_t. It is generated by (or generates) a conditionally Poisson particle system.

Two-Type, No Mutation Case

Let $E = \{0 \equiv \bigcirc, 1 \equiv \bullet\}$ with no mutation.

Then $X_t(\cdot \times \{1\}) \le X_t(\cdot \times E) = \ell_{\mathbb{R}}$, so let $X_t(dx \times \{1\}) = u_t(x)dx$.



Mueller-Tribe SPDE

$$\dot{u} = \frac{1}{2}\Delta u + |2\gamma u(1-u)|^{1/2}\,\dot{W}$$

That is, writing $u_t(f) = \int f(x)u_t(x)dx$, we have

$$u_t(f) = u_0(f) + \frac{1}{2}\int_0^t u_t(f'')ds + M_t^f$$

$$[M^f]_t = 2\gamma\int_0^t\int_{\mathbb{R}} f^2(x)u_s(x)\,(1-u_s(x))\,dxds$$

This is the X-measure for the infinite-density model with

- two types, no mutation;
- ind BMs at rate $\theta = 1$;
- ordered pair interacting at $\frac{\gamma}{2}dL_t^0(\hat{Z}_u - \hat{Z}_v)$.

Space-Time Scaling (Tribe 1995)

$$\dot{u} = \frac{1}{2}\Delta u + |2\gamma u(1-u)|^{1/2}\,\dot{W}$$

Start all zero to left of origin, all ones to right: $u_0 = 1_{[0,\infty)}$.

Define space-time scaling:

$$v_t^{(n)}(x) = u_{n^2t}(nx)$$

In the limit $n \to \infty$, there is a one-point interface whose motion converges to rate 1 Brownian motion.

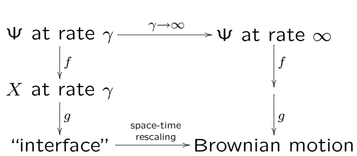
Another View of the Scaling

If $u_t(x)$ solves

$$\dot{u} = \frac{1}{2}\Delta u + |2\gamma u(1-u)|^{1/2}\,\dot{W}$$

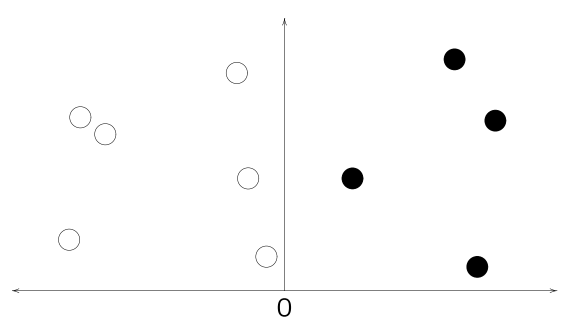
then $v_t^{(n)}(x) = u_{n^2t}(nx)$ solves

$$\dot{v}^{(n)} = \frac{1}{2}\Delta v^{(n)} + \left|2n\gamma v^{(n)}\left(1 - v^{(n)}\right)\right|^{1/2}\dot{W}$$



- Does γ-lookdown system converge to immediate-lookdown system as $\gamma \to \infty$?
- Is $g \circ f$ a "continuous" function of the particle system?

Immediate-Interaction Model



- at time 0, all ○ to left, all ● to right of origin;
- independent, rate 1 horizontal Brownian motions;
- immediate lookdowns: $\hat{Z}_u(t) = \hat{Z}_v(t)$ implies $\kappa_u(t) = \kappa_v(t)$.

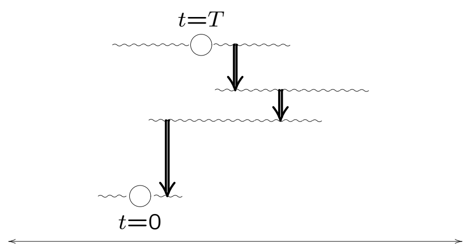
Reference: Donnelly, Evans, Fleischmann, Kurtz, and Zhou. "Continuum-sites stepping-stone models, coalescing exchangeable partitions, and random trees." *Annals of Probability*, 28(3):1063–1110, 2000.

Interface Persists

We never get a ○ to the right of a ● because of the...

Ancestral Trajectory of Particles

Track a particle at time $t = T$ backward to $t = 0$. Follow each lookdown.



Note that the type can't change: it can only change at a lookdown, and we follow those.

Interface Persists

Suppose at time $t = T$, there's a ○ to the right of a ●. Then:

- follow ancestral trajectories back to $t = 0$;
- at $t = 0$, ○ is left of ●, so they crossed;
- at crossing, there's a lookdown making them the same type, a contradiction.

Let's write:

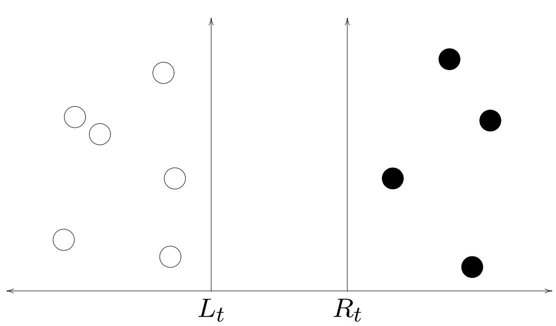
$$L_t = \sup\{\widehat{Z}_u(t) : \kappa_u(t) = 0 \equiv ○\}$$
$$R_t = \inf\{\widehat{Z}_u(t) : \kappa_u(t) = 1 \equiv ●\}$$

and note $L_t \leq R_t$.

Interface Is a Point

At each time t, particles are Poisson($\ell_{\mathbb{R}} \times \ell_{\mathbb{R}_+}$) in location/level space.

Suppose interface is larger than a point:



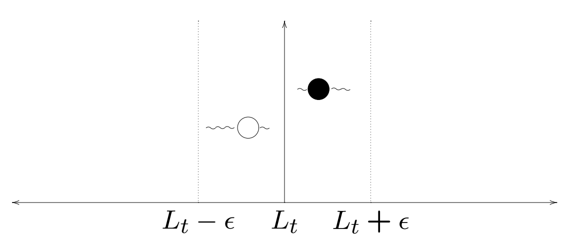
There are an infinite number of particles between L_t and R_t. So what are their types?

Lemma 1. *For all $a < b$ and $t \geq 0$ a.s., there are an infinite number of particles in (a, b) in a neighborhood of t.*

Interface Process Is Continuous

Lemma 2. *For all $a < b$ and $t \geq 0$ a.s., in a neighborhood of t, a particle confined to (a, b) does not change its type.*

Fix $t \geq 0$. For any $\epsilon > 0$, take intervals $(L_t - \epsilon, L_t)$ and $(L_t, L_t + \epsilon)$.



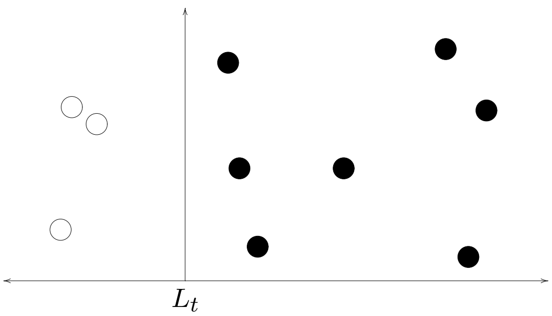
In t-nbd given by lemma, $|L_r - L_t| < \epsilon$.

Interface Inherits Structure

Lemma 3. *For all* $t \geq 0$,

$$\mathfrak{L}\big(\Psi_t \,\big|\, \mathfrak{F}_t^L\big) = \Pi_{L_t}$$

where $\Pi_x \equiv \text{Poisson}\left(\ell_{(-\infty,x)} \times \delta_0 + \ell_{[x,\infty)} \times \delta_1\right)$.



Proof of Lemma 3

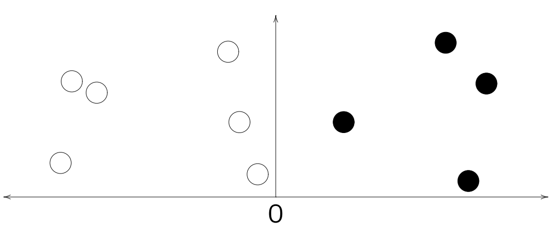
Enough to show that $\mathfrak{F}_t^L \subseteq \mathfrak{F}_t^X$ and
$X_t = \{\text{all } \bigcirc \text{ to left of } L_t, \text{ all } \bullet \text{ to right}\}$.

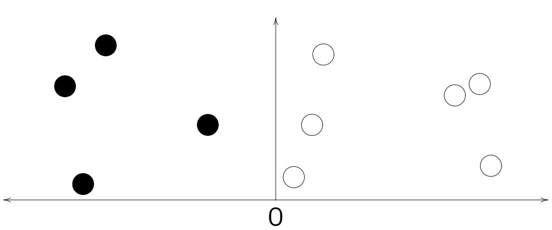
Then,

$$\begin{aligned} \mathsf{E}\left[h(\Psi_t) \,\middle|\, \mathfrak{F}_t^L\right] &= \mathsf{E}\left[\mathsf{E}\left[h(\Psi_t) \,\middle|\, \mathfrak{F}_t^X\right] \,\middle|\, \mathfrak{F}_t^L\right] \\ &= \mathsf{E}\left[\int h(\psi)\Pi_{X_t}(d\psi) \,\middle|\, \mathfrak{F}_t^L\right] \\ &= \int h(\psi)\Pi_{L_t}(d\psi) \end{aligned}$$

What Structure Is Inherited?

Space/type reversal of particle system has same distribution:

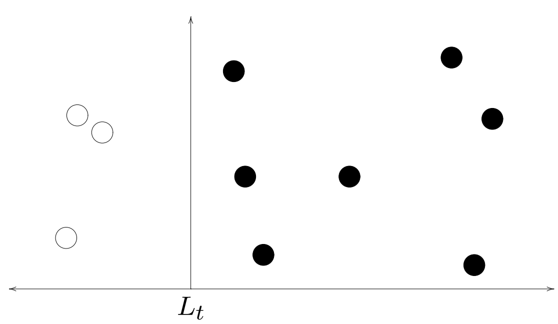




Therefore, $L_t =^d -L_t$.

What Structure Is Inherited?

Conditional distribution $\Psi_{t+\cdot} \,\big|\, \mathfrak{F}_t^L$ is same as $\Psi_\cdot$ shifted spatially.



Therefore, L_t has stationary, independent increments.

What is L?

The process L has:

- continuous paths;
- spatially symmetric distribution;
- stationary, independent increments.

So, L is a continuous, driftless Lévy process. Sounds like a Brownian motion!

This Technique Generalizes

Let the particle motion $(\hat{Z}, \hat{\mathsf{P}}^z)$ be any continuous, Borel process with dual (Z, P^z):

$$\int f P_t g dm = \int g \hat{P}_t f dm$$

for m having support on an interval of $\mathbb{R}$. Lay down initial particles using marginal location measure m.

Under some regularity conditions,

- ancestral trajectory is a copy of Z;
- initial interface persists as a point L_t;
- process L is continuous;
- $\mathfrak{L}\Big(\Psi_t \,\Big|\, \mathfrak{F}_t^L\Big) = \Pi_{L_t}$;
- L is a time-homogeneous Markov process.

Markov Property

Since $L_t \equiv h(\Psi_t)$ and Ψ_t is a time-homogeneous Markov process with semigroup $\mathbb{P}_t$,

$$\begin{aligned} \mathsf{E}\left[f(L_t) \,\middle|\, \mathfrak{F}_s^L\right] &= \mathsf{E}\left[\mathsf{E}\left[f \circ h(\Psi_t) \,\middle|\, \mathfrak{F}_s^\Psi\right] \,\middle|\, \mathfrak{F}_s^L\right] \\ &= \mathsf{E}\left[\mathbb{P}_{t-s} f \circ h(\Psi_s) \,\middle|\, \mathfrak{F}_s^L\right] \\ &= \int \mathbb{P}_{t-s} f \circ h(\psi) \Pi_{L_s}(d\psi) \end{aligned}$$

so L_t is time-homogeneous with semigroup,

$$\mathsf{P}_t^L f(x) = \int \mathbb{P}_t f \circ h(\psi) \Pi_x(d\psi)$$

A Duality Result

Theorem 4. $\mathsf{P}^l(L_t > y) = \mathsf{P}^y(Z_t < l)$

Proof. Follows immediately by a duality result in DEFKZ, 2000. □

Corollary 5. *For $\hat{Z}$ standard Brownian, the interface is standard Brownian.*

Corollary 6. *If Z has generator A and L has generator A_L, then*

$$\int f' A_L g + \int g' A f = 0$$

General Diffusion

A general diffusion with generator

$$\hat{A}f(x) = \frac{a(x)}{2}f''(x) + b(x)f'(x)$$

is "usually" self-dual with respect to its speed measure

$$m(dx) = m_0 e^{2\int_0^x \frac{b(y)}{a(y)}dy} a^{-1}(x)dx$$

So, by Corollary 6,

$$A_L g(x) = \frac{a(x)}{2}g''(x) + \left(\frac{a'(x)}{2} - b(x)\right) g'(x)$$

In particular, if $a(x) \equiv a > 0$, general $b(x)$ then A_L is diffusion with same diffusion rate and opposite drift.

Interesting Examples

Ornstein-Uhlenbeck

- $m(dx) = e^{-x^2/2}dx$
- Generators:

$$\hat{A}f(x) = \frac{1}{2}f''(x) - \frac{1}{2}xf'(x)$$
$$\hat{A}_L g(x) = \frac{1}{2}g''(x) + \frac{1}{2}xg'(x)$$

- Processes:

$$Z_t = e^{-t/2}\left(Z_0 + B\left(e^t - 1\right)\right)$$
$$L_t = e^{t/2}\left(L_0 + B\left(1 - e^{-t}\right)\right)$$

- SPDEs:

$$Z_t = Z_0 + B_t - \int_0^t \frac{1}{2}Z_s ds$$
$$L_t = L_0 + B_t + \int_0^t \frac{1}{2}L_s ds$$

Interesting Examples

Stochastic Exponential

- $m(dx) = x^{-(2c+1)}dx$
- Generators:

$$\hat{A}f(x) = \frac{1}{2}x^2 f''(x) + \left(\frac{1}{2} - c\right) xf'(x)$$
$$\hat{A}_L g(x) = \frac{1}{2}x^2 g''(x) + \left(\frac{1}{2} + c\right) xg'(x)$$

- Processes:

$$Z_t = Z_0 e^{B_t - ct}$$
$$L_t = L_0 e^{B_t + ct}$$

- SPDEs:

$$Z_t = Z_0 + \int_0^t Z_s dB_s + \left(\frac{1}{2} - c\right)\int_0^t Z_s ds$$
$$L_t = L_0 + \int_0^t L_s dB_s + \left(\frac{1}{2} + c\right)\int_0^t L_s ds$$

Another Interesting Connection

The deterministic solution of

$$\dot{u} = Au$$
$$u(0,x) = 1_{x \geq L_0}$$

is $u(x,t) = \mathsf{P}^x(Z_t \geq 0) = \mathsf{P}^0(L_t \leq x)$.

That is $u_t(\cdot)$ is the distribution function of L_t.