An Ordered Particle Construction of the Fleming-Viot Process

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Moran Model (1958)

Genetic interpretation:

- finite population of individuals;
- each individual has a genetic "type";
- types mutate;
- reproduction through matched birth/death events.

Brownian Moran Model

Fix population size n, mutation rate $\theta > 0$, and reproduction rate $\lambda > 0$.

Let $(X_1(t), \ldots, X_n(t))$ be the types of the individuals on $\mathbb{R}^d$, and LTFBI:

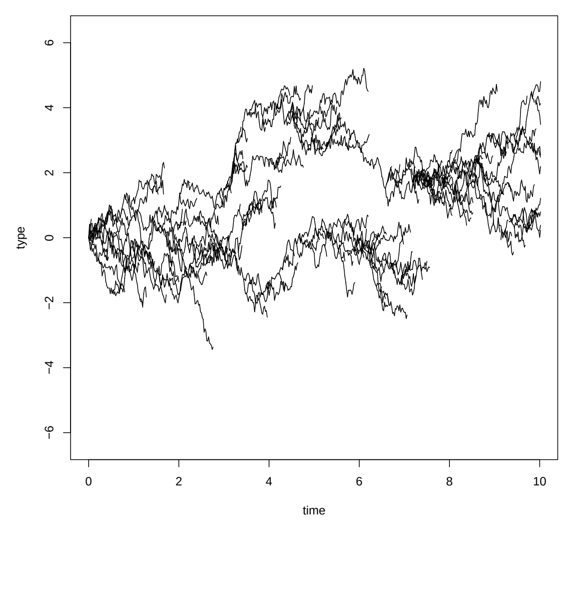
- $X_j(0)$ with distribution ν on $\mathbb{R}^d$;
- W_j standard Brownian motions;
- N_{ij} rate one Poisson counting processes.

For $1 \le j \le n$, let

$$X_j(t) = X_j(0) + \sqrt{\theta} W_j(t) + \sum_{\substack{1 \le i \le n \\ i \ne j}} \int_0^t \left(X_i(s-) - X_j(s-) \right) dN_{ij}\left(\tfrac{\lambda}{2} s\right)$$

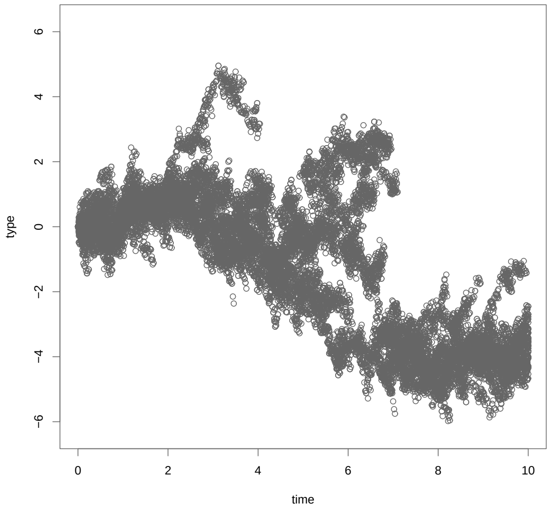
Brownian Moran Model

10-particle Moran model



Large-Population Limit

50-particle "large-population" approximation



Fleming-Viot Process, 1979

Showed that a similar model converged to a diffusion process by the "usual method."

Proved that:

1. sequence of prelimiting models is tight;
2. each limit point solves a martingale problem;
3. every solution to the MGP has the same distribution.

The Martingale Problem

Let Z_t be a $\mathfrak{M}_1(\mathbb{R}^d)$-valued process, and write $Z_t(\phi) = \int \phi \, dZ_t$.

1. $Z_0 = \nu$ with probability one;
2. For ϕ in a sufficiently rich class,

$$Z_t(\phi) = Z_0(\phi) + \int_0^t Z_s\left(\frac{\theta}{2}\Delta\phi\right) ds + M_t^\phi$$

where M_t^ϕ is a continuous, square integrable martingale with

$$\left\langle M^\phi, M^\psi \right\rangle_t = \lambda \int_0^t \Big(Z_s(\phi\psi) - Z_s(\phi)Z_s(\psi)\Big) ds$$

Comparison to Super Brownian Motion

In both cases, we have:

$$Z_t(\phi) = Z_0(\phi) + \int_0^t Z_s\left(\frac{\theta}{2}\Delta\phi\right) ds + M_t^\phi$$

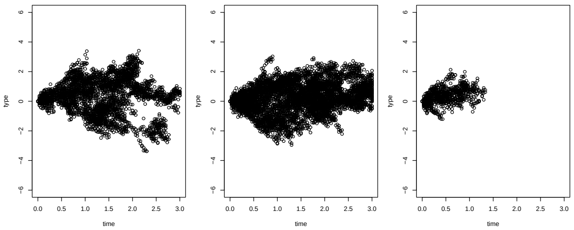
The difference is:

(FV) $$\left\langle M^\phi \right\rangle_t = \lambda \int_0^t \Big(Z_s(\phi^2) - (Z_s(\phi))^2\Big) ds$$

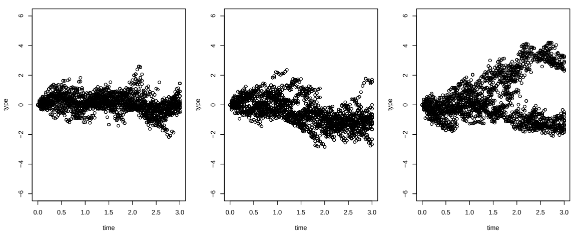
(SBM) $$\left\langle M^\phi \right\rangle_t = \lambda \int_0^t Z_s(\phi^2) ds$$

Comparison to Super Brownian Motion

Super Brownian Motion



Fleming-Viot Process



FV is SBM with global population control mechanism.

Ordered Particle Model

- Dawson & Hochberg, 1982
  - appeared implicitly in particle construction of moment measures;
  - used as tool to study support properties.
- Donnelly & Kurtz, 1996
  - more explicit construction;
  - used to study Fleming-Viot process with very general mutation;
  - used to study genealogy, ergodicity, sample-path properties.

Moran vs. Ordered Models

Usual Moran Model: for $1 \le j \le n$,

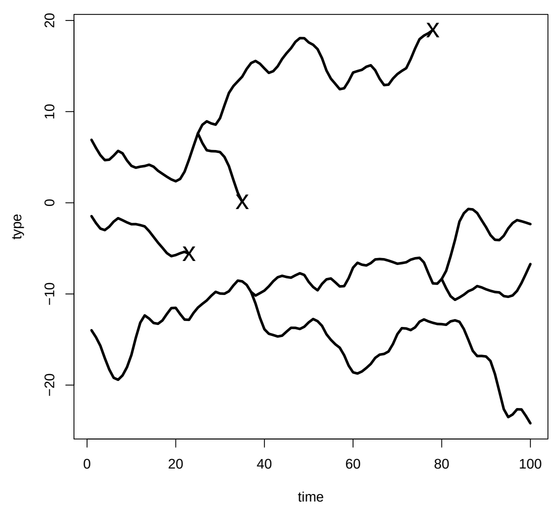
$$X_j(t) = X_j(0) + \sqrt{\theta} W_j(t) + \sum_{\substack{1 \le i \le n \\ i \ne j}} \int_0^t \Big(X_i(s-) - X_j(s-) \Big) \, dN_{ij}\left(\tfrac{\lambda}{2} s\right)$$

Ordered Model: for $j \in \mathbb{N}$,

$$X_j(t) = X_j(0) + \sqrt{\theta} W_j(t) + \sum_{i<j} \int_0^t \Big(X_i(s-) - X_j(s-) \Big) \, dN_{ij}(\lambda s)$$

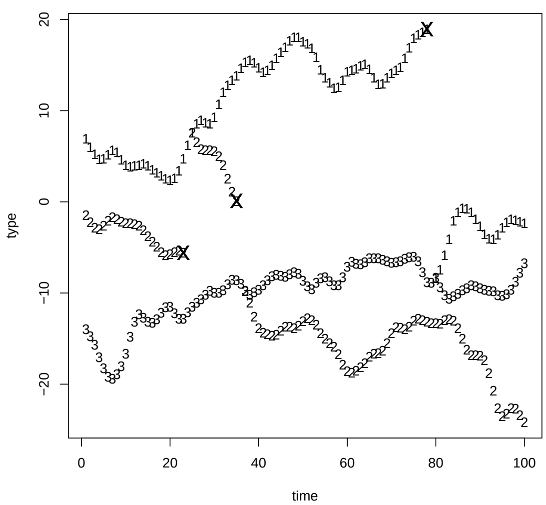
In the usual Moran model, indices have no special meaning. In the ordered model, they establish a "pecking order" or "rank."

This Ordering Is Special

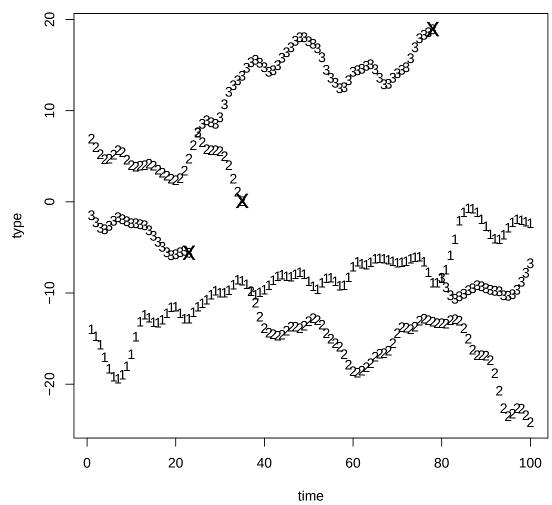


Q: Is this a realization of the Moran model with $n = 3$ or the first three particles of the (infinite) ordered model?

Answer: it's the Moran model!



No, it's the ordered model!



Coupling the Two Models

$$
\text{(I)} \quad \begin{cases} \forall j \in \mathbb{N}, \quad X_j(t) = X_j(0) + \sqrt{\theta} W_j(t) \\ \qquad\qquad + \displaystyle\sum_{i<j} \int_0^t \left(X_i(s-) - X_j(s-)\right) dN_{ij}(\lambda s) \end{cases}
$$

$$
\text{(II)} \quad \begin{cases} \forall j \leq n, \quad \tilde{X}_j(t) = \tilde{X}_j(0) + \sqrt{\theta} \tilde{W}_j(t) \\ \qquad\qquad + \displaystyle\sum_{\substack{1 \leq i \leq n \\ i \neq j}} \int_0^t \left(\tilde{X}_i(s-) - \tilde{X}_j(s-)\right) d\tilde{N}_{ij}\left(\tfrac{\lambda}{2} s\right) \\ \forall j > n, \quad \tilde{X}_j(t) = \tilde{X}_j(0) + \sqrt{\theta} \tilde{W}_j(t) \\ \qquad\qquad + \displaystyle\sum_{1 \leq i < j} \int_0^t \left(\tilde{X}_i(s-) - \tilde{X}_j(s-)\right) d\tilde{N}_{ij}(\lambda s) \end{cases}
$$

Coupling Theorem

Theorem 1. *For each n, there exists a probability space carrying models (I), (II), and a permutation-valued process*

$$\Sigma(\omega, t) \colon \{1, \ldots, n\} \leftrightarrow \{1, \ldots, n\}$$

with distribution

$$\mathrm{P}\Big(\Sigma_t = \sigma \;\Big|\; X_j(0), W_j(r), N_{ij}(r), r \leq t\Big) = \mathrm{P}\Big(\Sigma_t = \sigma \;\Big|\; \tilde{X}_j(0), \tilde{W}_j(r), \tilde{N}_{ij}(r), r \leq t\Big) = \frac{1}{n!}$$

for all $\sigma \in S_n$ such that

$$X_j(\omega, t) = \begin{cases} \tilde{X}_{\Sigma(\omega,t,j)}(\omega, t), & j \leq n; \\ \tilde{X}_j(\omega, t), & j > n. \end{cases}$$

for all $\omega \in \Omega$, $t \geq 0$.

Outline of Proof

Proof of Theorem 1. (B, 2002), (Donnelly & Kurtz, 1998).

- start with a random permutation $\Sigma(0)$, interpreting $\Sigma_j(t)$ (rather than j itself) as the particle's "rank" for $j \le n$;
- when two particles $i, j \le n$ interact, higher rank particle copies lower rank particle and—half the time—they swap ranks;
- if you pretend the ranks aren't there, it looks like model (II), but if you index particles by their (changing!) ranks, it looks like model (I).

□

Consequences of Theorem

For each fixed n,

1. Define empirical measures
$$Z_t^n = \frac{1}{n}\sum_{j=1}^n \delta_{X_j(t)} \qquad \tilde{Z}_t^n = \frac{1}{n}\sum_{j=1}^n \delta_{\tilde{X}_j(t)}$$
Then $Z^n =^d \tilde{Z}^n \equiv Moran(n)$ as processes.

2. Define
$$\mathfrak{F}_t^n = \sigma\left\{Z_r^n, X_{n+1}(r), X_{n+2}(r), \ldots : r \le t\right\}$$
Then, for all $t \ge 0$ and $\sigma \in S_n$,
$$\begin{aligned}\mathfrak{L}\left[(X_{\sigma_1}(t), \ldots, X_{\sigma_n}(t)) \mid \mathfrak{F}_t^n\right] \\ = \mathfrak{L}\left[(X_1(t), \ldots, X_n(t)) \mid \mathfrak{F}_t^n\right]\end{aligned}$$

Empirical Measure

Define empirical measure
$$Z_t^n = \frac{1}{n}\sum_{j=1}^n \delta_{X_j(t)}$$
and write
$$Z_t^n(\phi) = \frac{1}{n}\sum_{j=1}^n \phi(X_j(t))$$

Why might $Z_t^n(\phi)$ be of interest?

A Backward Martingale

Define
$$\mathfrak{F}_t^n = \sigma\left\{Z_r^n, X_{n+1}(r), X_{n+2}(r), \ldots : r \le t\right\}$$
and write $\mathfrak{F}_t^\infty = \bigcap_{n\ge 1}\mathfrak{F}_t^n$.

Lemma 2 (MGBCT). *If* $\mathsf{E}\,|X|^p < \infty$ *for some* $p \ge 1$, *then*
$$\mathsf{E}\Big[X \;\Big|\; \mathfrak{F}_t^n\Big] \to \mathsf{E}\Big[X \;\Big|\; \mathfrak{F}_t^\infty\Big]$$
almost surely and in L_p.

Limiting Empirical Measure

Lemma 3.

$$\mathsf{E}\Big[\phi(X_1(t)) \;\Big|\; \mathfrak{F}^n_t\Big] = Z^n_t(\phi)$$

Proof. For all $1 \le j \le n$,

$$\begin{aligned}\mathsf{E}\Big[\phi(X_1(t)) \;\Big|\; \mathfrak{F}^n_t\Big] &= \mathsf{E}\Big[\phi(X_j(t)) \;\Big|\; \mathfrak{F}^n_t\Big] \\ &= \mathsf{E}\left[\frac{1}{n}\sum_{j=1}^{n} \phi(X_j(t)) \;\middle|\; \mathfrak{F}^n_t\right] \\ &= Z^n_t(\phi)\end{aligned}$$

□

Therefore, for bounded ϕ,

$$Z^n_t(\phi) \to Z_t(\phi) \equiv \mathsf{E}\Big[\phi(X_1(t)) \;\Big|\; \mathfrak{F}^\infty_t\Big]$$

almost surely and in L_p for all $p \ge 1$.

Martingale Problem

Since

$$X_1(t) = X_1(0) + \sqrt{\theta} W_1(t)$$

by Itô, we have

$$\begin{aligned}\phi(X_1(t)) - \phi(X_1(0)) - \frac{\theta}{2}\int_0^t \phi''(X_1(s))ds \\ = \sqrt{\theta}\int_0^t \phi'(X_1(s))dW_1(s)\end{aligned}$$

is an $\{\mathfrak{F}^1_t\}$-martingale, and so

$$\begin{aligned}\mathsf{E}\big[\phi(X_1(t)) \mid \mathfrak{F}^\infty_t\big] - \mathsf{E}\big[\phi(X_1(0)) \mid \mathfrak{F}^\infty_0\big] \\ - \frac{\theta}{2}\int_0^t \mathsf{E}\big[\phi''(X_1(s)) \mid \mathfrak{F}^\infty_s\big]ds\end{aligned}$$

is an $\{\mathfrak{F}^\infty_t\}$-martingale. Therefore,

$$Z_t(\phi) = Z_0(\phi) + \int_0^t Z_s\left(\tfrac{\theta}{2}\phi''\right) ds + M^\phi_t$$

for some $\{\mathfrak{F}^\infty_t\}$-martingale M^ϕ.

Quadratic Variation

By Itô,

$$\begin{aligned}M^{\phi,n}_t &= Z^n_t(\phi) - Z^n_0(\phi) - \int_0^t Z^n_s\left(\tfrac{\theta}{2}\phi''\right) ds \\ &= \frac{\sqrt{\theta}}{n}\sum_{j=1}^{n}\int_0^t \phi'(X_j(s))dW_j(s) \\ &\quad + \frac{1}{n}\sum_{1\le i<j\le n}\int_0^t \left(\phi(X_i(s-)) - \phi(X_j(s-))\right) dN_{ij}(\lambda s)\end{aligned}$$

so

$$\begin{aligned}M^\phi_t &= Z_t(\phi) - Z_0(\phi) - \int_0^t Z_s\left(\tfrac{\theta}{2}\phi''\right) ds \\ &= \lim_{n\to\infty}\left\{Z^n_t(\phi) - Z^n_0(\phi) - \int_0^t Z^n_s\left(\tfrac{\theta}{2}\phi''\right) ds\right\} \\ &= \lim_{n\to\infty} M^{\phi,n}_t\end{aligned}$$

Quadratic Variation

Theorem 4.

1. *$M^{\phi,n}$ is an $\{\mathfrak{F}^n_t\}$-martingale;*
2. *For all t,*

   $$\left\langle M^{\phi,n}, M^{\psi,n}\right\rangle_t \xrightarrow{L^1} A_t$$

   for

   $$A_t \equiv \lambda\int_0^t \left(Z_s(\phi\psi) - Z_s(\phi)Z_s(\psi)\right) ds$$

3. *$\left\langle M^\phi, M^\psi\right\rangle = A$.*

Final Martingale Problem

The process Z_t satisfies

$$Z_t(\phi) = Z_0(\phi) + \int_0^t Z_s\left(\frac{\theta}{2}\phi''\right) ds + M_t^{\phi}$$

for M^{ϕ} a continuous, square integrable martingale such that

$$\left\langle M^{\phi}, M^{\psi} \right\rangle_t = \lambda \int_0^t \left(Z_s(\phi\psi) - Z_s(\phi)Z_s(\psi)\right) ds$$