MATH 263 ASSIGNMENT 9 SOLUTIONS

1) Let $\vec{F} = (x - yz)\hat{\imath} + (y + xz)\hat{\jmath} + (z + 2xy)\hat{k}$ and let

S_1 be the portion of the cylinder $x^2 + y^2 = 2$ that lies inside the sphere $x^2 + y^2 + z^2 = 4$

S_2 be the portion of the sphere $x^2 + y^2 + z^2 = 4$ that lies outside the cylinder $x^2 + y^2 = 2$

V be the volume bounded by S_1 and S_2

Compute

a) $\iint_{S_1} \vec{F} \cdot \hat{n}\, dS$ with $\hat{n}$ pointing inward

b) $\iiint_V \vec{\nabla} \cdot F\, dV$

c) $\iint_{S_2} \vec{F} \cdot \hat{n}\, dS$ with $\hat{n}$ pointing outward

Use the divergence theorem to answer at least one of parts (a), (b) and (c).

Solution. Observe that $\vec{\nabla} \cdot \vec{F} = 3$. So

$$\iiint_V \vec{\nabla} \cdot F\, dV = \iiint_V 3\, dV$$

The horizontal cross-section of V at height z is a washer with outer radius $\sqrt{4 - z^2}$ (determined by the equation of the sphere) and inner radius $\sqrt{2}$ (determined by the equation of the cylinder). So the cross-section has area $\pi\big(4 - z^2\big) - \pi 2 = \pi\big(2 - z^2\big)$. On the intersection of the sphere and cylinder $z^2 = 4 - 2 = 2$ so

$$\iiint_V \vec{\nabla} \cdot F\, dV = 3\int_{-\sqrt{2}}^{\sqrt{2}} \pi\big(2 - z^2\big)\, dz = 6\pi \int_0^{\sqrt{2}} \big(2 - z^2\big)\, dz = 6\pi\big(2\sqrt{2} - \tfrac{2^{3/2}}{3}\big) = \boxed{8\sqrt{2}\pi}$$

On the cylindrical surface, using (surprise!) cylindrical coordinates,

$$\begin{aligned}
\hat{n} &= -\big(\cos\theta \hat{\boldsymbol{\imath}} + \sin\theta \hat{\boldsymbol{\jmath}}\big) \\
dS &= \sqrt{2}\, d\theta\, dz \\
\vec{F} \cdot \hat{n} &= \sqrt{2}\big(\cos\theta - z\sin\theta\big)\big(-\cos\theta\big) + \sqrt{2}\big(\sin\theta + z\cos\theta\big)\big(-\sin\theta\big) = -\sqrt{2}
\end{aligned}$$

so

$$\iint_{S_1} \vec{F} \cdot \hat{n}\, dS = -2\int_{-\sqrt{2}}^{\sqrt{2}} dz \int_0^{2\pi} d\theta = \boxed{-8\sqrt{2}\pi}$$

By the divergence theorem

$$\iint_{S_2} \vec{F} \cdot \hat{n}\, dS = \iiint_V \vec{\nabla} \cdot F\, dV - \iint_{S_1} \vec{F} \cdot \hat{n}\, dS = \boxed{16\sqrt{2}\pi}$$

2) Evaluate the integral $\iiint_S \vec{F} \cdot \hat{n}\, dS$, where $\vec{F} = (x, y, 1)$ and S is the surface $z = 1 - x^2 - y^2$, for $x^2 + y^2 \le 1$, by two methods.

a) First, by direct computation of the surface integral.

b) Second, by using the divergence theorem.

Solution. a) Let $G(x, y, z) = x^2 + y^2 + z$. Then

$$\begin{aligned}
\hat{n}\, dS &= \tfrac{\vec{\nabla} G}{\vec{\nabla} G \cdot \hat{\mathbf{k}}}\, dx\, dy = \tfrac{2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}}{1}\, dx\, dy = (2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})\, dx\, dy \\
\vec{F} \cdot \hat{n}\, dS &= \big[x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big] \cdot \big[2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big]\, dx\, dy = \big[2x^2 + 2y^2 + 1\big]\, dx\, dy
\end{aligned}$$

Switching to polar coordinates

$$\iiint_S \vec{F} \cdot \hat{n}\, dS = \int_0^1 dr\ r \int_0^{2\pi} d\theta\ (2r^2 + 1) = 2\pi\Big[\tfrac{1}{2}r^4 + \tfrac{1}{2}r^2\Big]_0^1 = \boxed{2\pi}$$

b) Call the solid $0 \le z \le 1 - x^2 - y^2$, V. Let D denote the bottom surface of V. The disk D has radius 1, area π, $z = 0$ the outward normal $-\hat{\mathbf{k}}$, so that

$$\iint_D \vec{F} \cdot \hat{n}\, dS = -\iint_D \vec{F} \cdot \hat{\mathbf{k}}\, dx\, dy = -\iint_D dx\, dy = -\pi$$

As

$$\vec{\nabla} \cdot \vec{F} = \tfrac{\partial}{\partial x}(x) + \tfrac{\partial}{\partial y}(y) + \tfrac{\partial}{\partial z}(1) = 2$$

the divergence theorem gives

$$\begin{aligned}\iint_S \vec{F} \cdot \hat{n}\, dS &= \iiint_V \vec{\nabla} \cdot \vec{F}\ dV - \iint_D \vec{F} \cdot \hat{n}\, dS = \iiint_V 2\ dV - (-\pi) \\ &= \pi + 2\int_0^1 dz \iint_{x^2+y^2 \le 1-z} dxdy = \pi + 2\int_0^1 dz\ \pi(1-z) \\ &= \pi + 2\pi\Big[z - \tfrac{1}{2}z^2\Big]_0^1 = \boxed{2\pi}\end{aligned}$$

3a) By applying the divergence theorem to $\vec{F} = \phi\vec{a}$, where $\vec{a}$ is an arbitrary constant vector, show that

$$\iiint_V \vec{\nabla}\phi\, dV = \iint_{\partial V} \phi\hat{n}\, dS$$

b) Show that the centroid $(\bar{x}, \bar{y}, \bar{z})$ of a solid V is given by

$$(\bar{x}, \bar{y}, \bar{z}) = \tfrac{1}{2\,\mathrm{vol}(V)} \iint_{\partial V} (x^2 + y^2 + z^2)\, \hat{n}\, dS$$

Solution. a) The divergence of $\phi\vec{a}$ is $\vec{\nabla}\phi \cdot \vec{a}$. So, by the divergence theorem,

$$\iint_{\partial V} \phi\vec{a} \cdot \hat{n}\, dS = \iiint_V \vec{\nabla}\phi \cdot \vec{a}\ dV \implies \Big[\iint_{\partial V} \phi\hat{n}\, dS - \iiint_V \vec{\nabla}\phi\, dV\Big] \cdot \vec{a} = 0$$

This is true for all vectors $\vec{a}$. So

$$\iint_{\partial V} \phi\hat{n}\, dS - \iiint_V \vec{\nabla}\phi\, dV = 0$$

b) By part a, with $\phi = x^2 + y^2 + z^2$,

$$\tfrac{1}{2\,\mathrm{vol}(V)} \iint_{\partial V} (x^2 + y^2 + z^2)\, \hat{n}\, dS = \tfrac{1}{2\,\mathrm{vol}(V)} \iiint_V (2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{\mathbf{k}})\, dV = (\bar{x}, \bar{y}, \bar{z})$$

4) Find the flux of $\vec{F} = (y + xz)\hat{\imath} + (y + yz)\hat{\jmath} - (2x + z^2)\hat{\mathbf{k}}$ upward through the first octant part of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution. Let $V = \big\{\ (x, y, z)\ \big|\ x^2 + y^2 + z^2 \le a^2, x \ge 0, y \ge 0,\ z \ge 0\ \big\}$. The ∂V consists of an $x = 0$ face, a $y = 0$ face, a $z = 0$ face and the first octant part of the sphere. Call the latter S. Then

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iiint_V \big[z + 1 + z - 2z\big]\, dV = \iiint_V dV = \tfrac{1}{8}\tfrac{4}{3}\pi a^3 = \tfrac{1}{6}\pi a^3$$

$$\iint_{\substack{x=0\\ \text{face}}} \vec{F} \cdot (-\hat{\imath})\, dy\, dz = \iint_{\substack{x=0\\ \text{face}}} (-y)\, dy\, dz = -\int_0^a dr\, r \int_0^{\pi/2} d\theta\ r\sin\theta = -\int_0^a r^2\, dr = -\tfrac{a^3}{3}$$

$$\iint_{\substack{y=0\\ \text{face}}} \vec{F} \cdot (-\hat{\jmath})\, dx\, dz = 0$$

$$\iint_{\substack{z=0\\ \text{face}}} \vec{F} \cdot (-\hat{\mathbf{k}})\, dx\, dy = \iint_{\substack{z=0\\ \text{face}}} (2x)\, dx\, dy = \tfrac{2a^3}{3}$$

By the divergence theorem

$$\iint_s \vec{F}\cdot\hat{n}\,dx\,dy = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV - \iint_{\substack{x=0\\ \text{face}}} \vec{F}\cdot(-\hat{\boldsymbol{\imath}})\,dy\,dz - \iint_{\substack{y=0\\ \text{face}}} \vec{F}\cdot(-\hat{\boldsymbol{\jmath}})\,dx\,dz - \iint_{\substack{z=0\\ \text{face}}} \vec{F}\cdot(-\hat{\mathbf{k}})\,dx\,dy = \boxed{\big[\tfrac{\pi}{6}-\tfrac{1}{3}\big]a^3}$$

5) Let $\vec{E}(\vec{r})$ be the electric field due to a charge configuration that has density $\rho(\vec{r})$. Gauss' law states that, if V is any solid in $\mathbb{R}^3$ with surface ∂V, then the electric flux

$$\iint_{\partial V} \vec{E}\cdot\hat{n}\,dS = 4\pi Q \qquad \text{where} \qquad Q = \iiint_V \rho\ dV$$

is the total charge in V. Here, as usual, $\hat{n}$ is the outward pointing unit normal to ∂V. Show that

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = 4\pi\rho(\vec{r})$$

for all $\vec{r}$ in $\mathbb{R}^3$. This is one of Maxwell's equations. Assume that $\vec{\nabla}\cdot\vec{E}(\vec{r})$ and $\rho(\vec{r})$ are well–defined and continuous everywhere.

Solution. By the divergence theorem

$$\iint_{\partial V} \vec{E}\cdot\hat{n}\,dS = \iiint_V \vec{\nabla}\cdot\vec{E}\,dV$$

So by Gauss' law

$$\iiint_V \vec{\nabla}\cdot\vec{E}\,dV = 4\pi\iiint_V \rho\ dV \qquad \Rightarrow \qquad \iiint_V \big[\vec{\nabla}\cdot\vec{E} - 4\pi\rho\big]\,dV = 0$$

This is true for all solids V for which the divergence theorem applies. If there were some point in $\mathbb{R}^3$ for which $\vec{\nabla}\cdot\vec{E}-4\pi\rho$ were, say, strictly bigger than zero, then, by continuity, we could find a ball B_ϵ centered on that point with $\vec{\nabla}\cdot\vec{E}-4\pi\rho > 0$ everywhere on B_ϵ. This would force $\iiint_{B_\epsilon}\big[\vec{\nabla}\cdot\vec{E}-4\pi\rho\big]\,dV > 0$, which violates $\iiint_V\big[\vec{\nabla}\cdot\vec{E}-4\pi\rho\big]\,dV = 0$ with V set equal to B_ϵ. Hence $\vec{\nabla}\cdot\vec{E}-4\pi\rho$ must be zero everywhere.

6) Evaluate, both by direct integration and by Stokes' Theorem, $\oint_C (z\,dx + x\,dy + y\,dz)$ where C is the circle $x+y+z=0$, $x^2+y^2+z^2=1$. Orient C so that its projection on the xy–plane is counterclockwise.

Solution. The projection of C on the xy–plane is $x^2+y^2+(-x-y)^2=1$ or $2x^2+2xy+2y^2=1$ or $\frac{3}{2}(x+y)^2+\frac{1}{2}(x-y)^2=1$. Hence we may parametrize the curve using

$$x+y=\sqrt{\tfrac{2}{3}}\cos\theta,\ x-y=-\sqrt{2}\sin\theta \ \Rightarrow \begin{cases} x(\theta)=\frac{1}{\sqrt6}\cos\theta-\frac{1}{\sqrt2}\sin\theta\\ y(\theta)=\frac{1}{\sqrt6}\cos\theta+\frac{1}{\sqrt2}\sin\theta\\ z(\theta)=-\frac{2}{\sqrt6}\cos\theta\end{cases} \text{and} \begin{cases} x'(\theta)=-\frac{1}{\sqrt6}\sin\theta-\frac{1}{\sqrt2}\cos\theta\\ y'(\theta)=-\frac{1}{\sqrt6}\sin\theta+\frac{1}{\sqrt2}\cos\theta\\ z'(\theta)=\frac{2}{\sqrt6}\sin\theta\end{cases}$$

The sign in $x-y=-\sqrt{2}\sin\theta$ has been chosen to make the projected motion counterclockwise. The check this, observe that at $\theta=0$, $(x,y)=\frac{1}{\sqrt6}(1,1)$ and $\big(\frac{dx}{d\theta},\frac{dy}{d\theta}\big)=\frac{1}{\sqrt2}(-1,1)$, which is up and to the left. This integral is of the form $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=z\hat{\boldsymbol{\imath}}+x\hat{\boldsymbol{\jmath}}+y\hat{\mathbf{k}}$ and C is curve parametrized above. Hence

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \vec{F}\big(\vec{r}(\theta)\big)\cdot\vec{r}\,'(\theta)\ d\theta\\ &= \int_0^{2\pi}\Big[-\tfrac{2}{\sqrt6}\cos\theta\big(-\tfrac{1}{\sqrt6}\sin\theta-\tfrac{1}{\sqrt2}\cos\theta\big)\\ &\qquad + \big(\tfrac{1}{\sqrt6}\cos\theta-\tfrac{1}{\sqrt2}\sin\theta\big)\big(-\tfrac{1}{\sqrt6}\sin\theta+\tfrac{1}{\sqrt2}\cos\theta\big)\\ &\qquad + \big(\tfrac{1}{\sqrt6}\cos\theta+\tfrac{1}{\sqrt2}\sin\theta\big)\big(\tfrac{2}{\sqrt6}\sin\theta\big)\Big]\ d\theta\\ &= \int_0^{2\pi}\Big[\tfrac{3}{\sqrt{12}}\cos^2\theta+\tfrac{3}{\sqrt{12}}\sin^2\theta+\big(\tfrac13-\tfrac16-\tfrac12+\tfrac13\big)\sin\theta\cos\theta\Big]\ d\theta\\ &= \tfrac{6\pi}{\sqrt{12}} = \boxed{\sqrt{3}\pi}\end{aligned}$$

Choose as S the portion of the plane $x + y + z = 0$ interior to the sphere. Then $\hat{n} = \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{\mathbf{k}})$ and $\vec{\nabla} \times \vec{F} = \hat{\imath} + \hat{\jmath} + \hat{\mathbf{k}}$ so, by Stokes' Theorem,

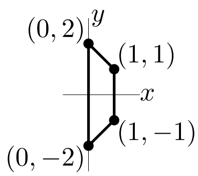
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n}\ dS = \iint_S (\hat{\imath} + \hat{\jmath} + \hat{\mathbf{k}}) \cdot \tfrac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{\mathbf{k}})\ dS = \sqrt{3} \iint_S dS = \boxed{\sqrt{3}\pi}$$

since S is a circle of radius 1.

7) Evaluate $\oint_C (x \sin y^2 - y^2)dx + (x^2 y \cos y^2 + 3x)dy$ where C is the counterclockwise boundary of the trapezoid with vertices $(0, -2)$, $(1, -1)$, $(1, 1)$ and $(0, 2)$.

Solution. By Green's theorem (or Stokes' theorem)

$$\oint_C (x \sin y^2 - y^2)dx + (x^2 y \cos y^2 + 3x)dy = \iint_T \left(\tfrac{\partial}{\partial x}(x^2 y \cos y^2 + 3x) - \tfrac{\partial}{\partial y}(x \sin y^2 - y^2)\right) dx\, dy$$



$$= \iint_T \left(2xy \cos y^2 + 3 - 2xy \cos y^2 + 2y\right) dx\, dy$$

$$= \iint_T \left(3 + 2y\right) dx\, dy$$

The integral of $2y$ vanishes because the domain of integration is invariant under $y \to -y$. The other integral is 3 times the area of the trapezoid, which is its width (1) times the average of its heights $(\frac{1}{2}[2 + 4])$. So $\boxed{\oint_C (x \sin y^2 - y^2)dx + (x^2 y \cos y^2 + 3x)dy = 9}$.

8) Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = ye^x \hat{\imath} + (x + e^x)\hat{\jmath} + z^2 \hat{\mathbf{k}}$ and C is the curve

$$\vec{r}(t) = (1 + \cos t)\hat{\imath} + (1 + \sin t)\hat{\jmath} + (1 - \sin t - \cos t)\hat{\mathbf{k}}$$

Solution. $\vec{\nabla} \times \vec{F} = \left(1 + e^x - e^x\right)\hat{\mathbf{k}}$, so by Stokes' Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \hat{\mathbf{k}} \cdot d\vec{S}$$

where S is the intersection of $x + y + z = 3$ with $(x - 1)^2 + (y - 1)^2 \le 1$. Now $\iint_S \hat{\mathbf{k}} \cdot d\vec{S}$ is the area of the projection of S on the xy–plane. This projection is the circle of radius 1 centred on $(1, 1)$, which has area π. So $\boxed{\oint_C \vec{F} \cdot d\vec{r} = \pi}$.

9) Let C be the intersection of $x+2y-z = 7$ and $x^2-2x+4y^2 = 15$. The curve C is oriented counterclockwise when viewed from high on the z–axis. Let

$$\vec{F} = \left(x^3 e^{-x} + yz\right)\hat{\imath} + \left(\tfrac{\sin y}{y} + \sin z - x^2\right)\hat{\jmath} + \left(xy + y \cos z\right)\hat{\mathbf{k}}$$

Evaluate $\oint_C \vec{F} \cdot d\vec{r}$.

Solution. By Stokes' Theorem and the observation that $\vec{\nabla} \times \vec{F} = x\hat{\imath} - (z + 2x)\hat{\mathbf{k}}$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n}\, dS \quad \text{where } S \text{ is the part of } x + 2y - z = 7 \text{ inside } (x - 1)^2 + 4y^2 = 16$$

$$= \iint_S [x\hat{\imath} - (z + 2x)\hat{\mathbf{k}}]\Big|_{z=-7+x+2y} \cdot (-1, -2, 1)\, dx\, dy$$

$$= \iint_S [7 - 4x - 2y]\, dx\, dy$$

$$= (\text{area of ellipse with semi–axes } a = 4,\ b = 2)[7 - 4\bar{x} - 2\bar{y}]$$

$$= \pi \times 4 \times 2[7 - 4 \times 1 - 2 \times 0] = \boxed{24\pi}$$

10) Consider $\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS$ where S is the portion of the sphere $x^2+y^2+z^2=1$ that obeys $x+y+z\ge 1$, $\hat{n}$ is the upward pointing normal to the sphere and $\vec{F}=(y-z)\hat{\pmb{\imath}}+(z-x)\hat{\pmb{\jmath}}+(x-y)\hat{\mathbf{k}}$. Find another surface S' with the property that $\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS=\iint_{S'}(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS$ and evaluate $\iint_{S'}(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS$.

Solution. Let S' be the portion of $x+y+z=1$ that is inside the sphere $x^2+y^2+z^2=1$. Then $\partial S=\partial S'$, so, by Stokes' Theorem, (with $\hat{n}$ always the upward pointing normal)

$$\iint_{S'}(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS=\oint_{\partial S'}\vec{F}\cdot d\vec{r}=\oint_{\partial S}\vec{F}\cdot d\vec{r}=\iint_S(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS$$

As $\vec{\nabla}\times\vec{F}=-2(\hat{\pmb{\imath}}+\hat{\pmb{\jmath}}+\hat{\mathbf{k}})$ and, on S', $\hat{n}=\frac{1}{\sqrt{3}}(\hat{\pmb{\imath}}+\hat{\pmb{\jmath}}+\hat{\mathbf{k}})$

$$\iint_{S'}(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS=\iint_{S'}\big(-2\sqrt{3}\big)\,dS=-2\sqrt{3}\times\text{Area}(S')$$

S' is a circular disk. It's center (x_c,y_c,z_c) has to obey $x_c+y_c+z_c=1$. By symmetry, $x_c=y_c=z_c$, so $x_c=y_c=z_c=\frac{1}{3}$. Any point, like $(0,0,1)$, which satisfies both $x+y+z=1$ and $x^2+y^2+z^2=1$ is on the boundary of S'. So the radius of S' is $\big\|\big(\frac{1}{3},\frac{1}{3},\frac{1}{3}\big)-(0,0,1)\big\|=\big\|\big(\frac{1}{3},\frac{1}{3},-\frac{2}{3}\big)\big\|=\sqrt{\frac{2}{3}}$. So the area of S' is $\frac{2}{3}\pi$ and

$$\iint_{S'}(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS=-2\sqrt{3}\times\text{Area}(S')=\boxed{-\tfrac{4}{\sqrt{3}}\pi}$$