Math 263 Assignment #8 Solutions

1. For each of the following vector fields $\vec{\mathbf{F}}$, find $\operatorname{div}\vec{\mathbf{F}}$ and $\operatorname{curl}\vec{\mathbf{F}}$.

$$\text{(a) } \vec{\mathbf{F}}(x,y,z) = z\vec{\mathbf{j}} - y\vec{\mathbf{k}} \quad \text{(b) } \vec{\mathbf{F}}(x,y,z) = \left\langle x^2, y, z\right\rangle \quad \text{(c) } \vec{\mathbf{F}}(x,y,z) = \left\langle x+y, -y^2, -2z\right\rangle$$

(a)

$$\operatorname{div}\vec{\mathbf{F}} = \nabla \bullet \vec{\mathbf{F}} = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3 = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(-y) = 0+0+0 = 0$$

$$\operatorname{curl}\vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & z & -y \end{vmatrix} = \vec{\mathbf{i}}\left(\frac{\partial}{\partial y}(-y) - \frac{\partial}{\partial z}(z)\right) - \vec{\mathbf{j}}\left(\frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial z}(0)\right)$$

$$+\,\vec{\mathbf{k}}\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(0)\right) = \langle -2, 0, 0\rangle$$

(b)

$$\operatorname{div}\vec{\mathbf{F}} = \nabla \bullet \vec{\mathbf{F}} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 2x+1+1 = 2x+2$$

$$\operatorname{curl}\vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \vec{\mathbf{i}}\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right) - \vec{\mathbf{j}}\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x^2)\right)$$

$$+\,\vec{\mathbf{k}}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x^2)\right) = \langle 0, 0, 0\rangle$$

(c)

$$\operatorname{div}\vec{\mathbf{F}} = \nabla \bullet \vec{\mathbf{F}} = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(-2z) = 1-2y-2 = -2y-1$$

$$\operatorname{curl}\vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \vec{\mathbf{i}}\left(\frac{\partial}{\partial y}(-2z) - \frac{\partial}{\partial z}(-y^2)\right) - \vec{\mathbf{j}}\left(\frac{\partial}{\partial x}(-2z) - \frac{\partial}{\partial z}(x+y)\right)$$

$$+\,\vec{\mathbf{k}}\left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(x+y)\right) = \langle 0, 0, -1\rangle$$

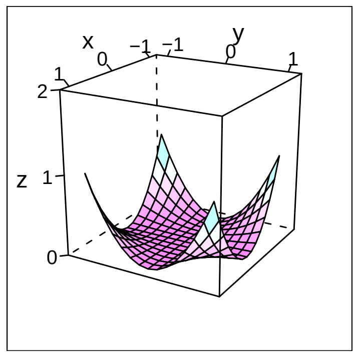
2. For each of the following oriented surfaces $\mathcal{S}$, (i) sketch $\mathcal{S}$, (ii) parametrize $\mathcal{S}$, (iii) find the vector and scalar area elements $d\vec{\mathbf{S}}$ and dS for your parametrization, (iv) calculate the indicated surface or flux integral.

   (a) $\mathcal{S}$ given by $z = x^2y^2$, $-1 \le x \le 1$, $-1 \le y \le 1$ oriented positive side upward. Calculate $\iint_{\mathcal{S}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}}$ for $\vec{\mathbf{F}} = x\vec{\mathbf{\imath}} + \vec{\mathbf{\jmath}} + z\vec{\mathbf{k}}$.

   (b) $\mathcal{S}$ surface of ellipsoid $4x^2 + 4y^2 + z^2 - 6z + 5 = 0$ oriented inward. Calculate surface area of $\mathcal{S}$.

   (c) $\mathcal{S}$ surface of intersection of sphere $x^2 + y^2 + z^2 \le 4$ and plane $z = 1$ oriented away from the origin. Calculate flux away from the origin of the electrical field $\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \dfrac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|^3}$.

(a)



Parametrizing $\mathcal{S}$ in x and y gives

$$\vec{\mathbf{r}}(x, y) = \langle x, y, x^2y^2 \rangle, \ -1 \le x \le 1, \ -1 \le y \le 1$$

The vector area element is given by

$$d\vec{\mathbf{S}} = \pm \left(\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right) dx\, dy = \pm \begin{pmatrix} \vec{\mathbf{\imath}} & \vec{\mathbf{\jmath}} & \vec{\mathbf{k}} \\ 1 & 0 & 2xy^2 \\ 0 & 1 & 2x^2y \end{pmatrix} dx\, dy = \pm \langle -2xy^2, -2x^2y, 1 \rangle \, dx\, dy$$

Since we want the positive side up, we choose the $d\vec{\mathbf{S}}$ with positive z-component: $d\vec{\mathbf{S}} = \langle -2xy^2, -2x^2y, 1 \rangle \, dx\, dy$. The scalar area element is

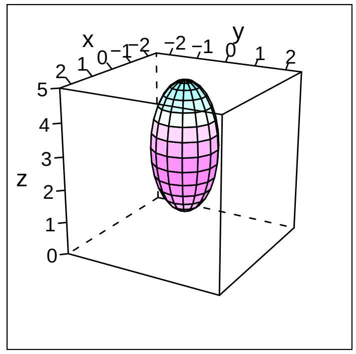
$$dS = \sqrt{4x^2y^4 + 4x^4y^2 + 1}\, dx\, dy$$

Finally, we have

$$\iint_{\mathcal{S}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = \iint_D \langle x, 1, x^2y^2 \rangle \bullet \langle -2xy^2, -2x^2y, 1 \rangle \, dx\, dy = \int_{-1}^{1} dx \int_{-1}^{1} dy [-x^2y^2 - 2x^2y]$$

$$= \int_{-1}^{1} dx \left(-\frac{x^2y^3}{3} - x^2y^2 \right) \Bigg|_{y=-1}^{y=1} = \int_{-1}^{1} \left(-\frac{2x^2}{3} \right) dx = -\frac{2}{9}x^3 \Bigg|_{x=-1}^{x=1} = -\frac{4}{9}$$

(b) Completing the square gives $4x^2 + 4y^2 + (z-3)^2 = 4$, so $\mathcal{S}$ is an ellipsoid centered at $(0, 0, 3)$ with semiaxes 1, 1, and 2:



In cylindrical coordinates, $\mathcal{S}$ consists of those points $[r, \theta, z]$ where $0 \le \theta \le 2\pi$, $1 \le z \le 5$, and $4r^2 + (z-3)^2 = 4$ or equivalently $r = \dfrac{1}{2}\sqrt{4-(z-3)^2}$. Therefore, we may parametrize it in θ and z as

$$\vec{\mathbf{r}}(\theta, z) = \left\langle \frac{1}{2}\sqrt{4-(z-3)^2}\cos\theta,\ \frac{1}{2}\sqrt{4-(z-3)^2}\sin\theta,\ z \right\rangle, \qquad 0 \le \theta \le 2\pi,\ 1 \le z \le 5$$

The vector area element is given by

$$\begin{aligned} d\vec{\mathbf{S}} = \pm\left(\frac{\partial r}{\partial\theta} \times \frac{\partial r}{\partial z}\right) d\theta\, dz &= \pm \begin{pmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -\frac{1}{2}\sqrt{4-(z-3)^2}\sin\theta & \frac{1}{2}\sqrt{4-(z-3)^2}\cos\theta & 0 \\ -\frac{1}{2}\dfrac{z-3}{\sqrt{4-(z-3)^2}}\cos\theta & -\frac{1}{2}\dfrac{z-3}{\sqrt{4-(z-3)^2}}\sin\theta & 2 \end{pmatrix} d\theta\, dz \\ &= \pm\left\langle \frac{1}{2}\sqrt{4-(z-3)^2}\cos\theta, \frac{1}{2}\sqrt{4-(z-3)^2}\sin\theta, \frac{1}{4}(z-3) \right\rangle d\theta\, dz \end{aligned}$$

We want the inward orientation, so we want the version that, say, points downward at $(x, y, z) = (0, 0, 5)$, thus:

$$d\vec{\mathbf{S}} = -\left\langle \frac{1}{2}\sqrt{4-(z-3)^2}\cos\theta, \frac{1}{2}\sqrt{4-(z-3)^2}\sin\theta, \frac{1}{4}(z-3) \right\rangle d\theta\, dz$$

The scalar area element is

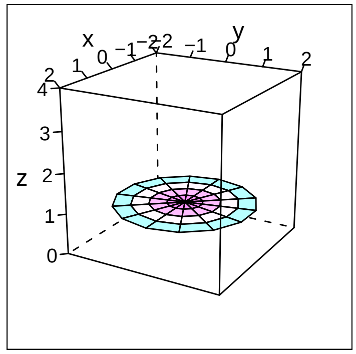
$$dS = \frac{1}{2}\sqrt{4 - \frac{3}{4}(z-3)^2}$$

giving a surface area

$$\begin{aligned} \iint_{\mathcal{S}} dS &= \int_0^{2\pi} d\theta \int_1^5 dz \left[\frac{1}{2}\sqrt{4-\frac{3}{4}(z-3)^2}\right] = \int_0^{2\pi} d\theta \int_{-\sqrt{3}}^{\sqrt{3}} du \frac{1}{\sqrt{3}}\sqrt{4-u^2} \\ &= \int_0^{2\pi} d\theta \frac{1}{\sqrt{3}}\left(\frac{u}{2}\sqrt{4-u^2} + 2\sin^{-1}\frac{u}{2}\right)\Big|_{u=-\sqrt{3}}^{u=\sqrt{3}} = \int_0^{2\pi} d\theta \frac{1}{\sqrt{3}}\left(\sqrt{3} + \frac{4\pi}{3}\right) \\ &= 2\pi\left(1 + \frac{4}{3\sqrt{3}}\pi\right) \end{aligned}$$

[Note: The substitution used above was $u = \frac{\sqrt{3}}{2}(z-3)$.]

(c)



The surface obeys $x^2 + y^2 \leq 3$ and $z = 1$, so parametrizing in the cylindrical coordinates r and θ, we have

$$\vec{\mathbf{r}}(r,\theta) = \langle r\cos\theta, r\sin\theta, 1\rangle\,, \quad 0 \leq r \leq \sqrt{3},\, 0 \leq \theta \leq 2\pi$$

giving a vector area element

$$d\vec{\mathbf{S}} = \pm\left(\frac{\partial\vec{\mathbf{r}}}{\partial r} \times \frac{\partial\vec{\mathbf{r}}}{\partial\theta}\right) dr\,d\theta = \pm\begin{pmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{pmatrix} dr\,d\theta = \pm\langle 0,0,r\rangle\,dr\,d\theta$$

For an orientation facing away from the origin (i.e., upward), we choose $d\vec{\mathbf{S}} = \langle 0,0,r\rangle\,dr\,d\theta$. The scalar area element is $dS = |d\vec{\mathbf{S}}| = r\,dr\,d\theta$.

Finally, the desired integral is

$$\iint_{\mathcal{S}} \vec{\mathbf{E}} \bullet d\vec{\mathbf{S}} = \int_0^{\sqrt{3}} dr \int_0^{2\pi} d\theta \left[\frac{\langle r\cos\theta, r\sin\theta, 1\rangle}{(r^2+1)^{3/2}} \bullet \langle 0,0,r\rangle\right] = \int_0^{\sqrt{3}} dr \int_0^{2\pi} d\theta \frac{r}{(r^2+1)^{3/2}}$$

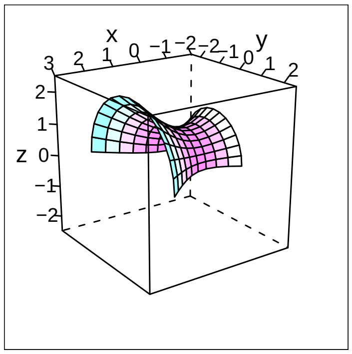
$$= 2\pi\int_0^{\sqrt{3}} \frac{r}{(r^2+1)^{3/2}} dr = \pi\int_1^4 u^{-3/2}du = -2\pi u^{-1/2}\Big|_{u=1}^{u=4} = \pi$$

[Note: The substitution used above was $u = r^2 + 1$.]

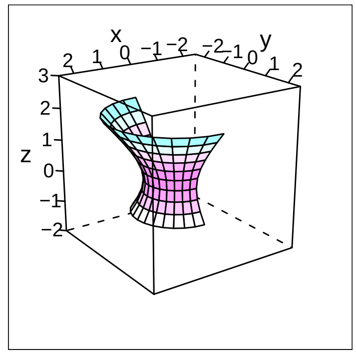
3. Let $\mathcal{S}$ be the portion of the surface $x^2 + 1 = y^2 + z^2$ bounded by the planes $x = -1$, $x = 2$, and lying above the xy-plane. Calculate the surface integral $\displaystyle\iint_{\mathcal{S}} z\sqrt{\frac{1+2x^2}{y^2+z^2}}\,dS$.

[Hint: You may find it helpful to restate the problem, exchanging the variables x, y, and z throughout to make the surface symmetric around the z-axis.]

This surface is the portion of the top half (that is, the portion above the plane $z = 0$) of a hyperboloid of one sheet centered around the x-axis that falls between $x = -1$ and $x = 2$. It looks like diagram (a) here:



(a)



(b)

However, if we restate the problem by swapping the variables x and z, then the new surface is the portion of $z^2 + 1 = y^2 + x^2$ between $z = -1$ and $z = 2$ that obeys $x \geq 0$, as shown in diagram (b), and we can calculate the rewritten surface integral $I = \displaystyle\iint_{\mathcal{S}} x\sqrt{\frac{1+2z^2}{y^2+x^2}}\,dS$ by first expressing the surface in cylindrical coordinates.

Note: For the remainder of this solution, we'll be working with the rewritten versions.

Note that the cylindrical coordinate $r = \sqrt{x^2+y^2}$ satisfies $r^2 = z^2 + 1$ on the surface $\mathcal{S}$, so $\mathcal{S}$ can be expressed in cylindrical coordinates as

$$\mathcal{S} = \left\{ \left[\sqrt{1+z^2}, \theta, z\right] : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ -1 \leq z \leq 2 \right\}$$

giving a parametrization

$$\vec{\mathbf{r}}(\theta, z) = \left\langle \sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, z \right\rangle, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ -1 \leq z \leq 2$$

The surface area element dS for this parametrization is then given by

$$\begin{aligned} dS = \left|\frac{\partial\vec{\mathbf{r}}}{\partial\theta} \times \frac{\partial\vec{\mathbf{r}}}{\partial z}\right| d\theta\,dz &= \left\| \begin{matrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -\sqrt{1+z^2}\sin\theta & \sqrt{1+z^2}\cos\theta & 0 \\ \frac{z}{\sqrt{1+z^2}}\cos\theta & \frac{z}{\sqrt{1+z^2}}\sin\theta & 1 \end{matrix} \right\| d\theta\,dz \\ &= \left|\left\langle \sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, -z \right\rangle\right| d\theta\,dz = \sqrt{1+2z^2}\,d\theta\,dz \end{aligned}$$

The integral may now be evaluated as follows:

$$\begin{aligned} I &= \int_{-1}^{2} dz \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \left[\left(\sqrt{1+z^2}\cos\theta\right) \sqrt{\frac{1+2z^2}{1+z^2}} \sqrt{1+2z^2} \right] = \int_{-1}^{2} dz \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \left[(1+2z^2)\cos\theta\right] \\ &= \int_{-1}^{2} (1+2z^2)\sin\theta\Big|_{\theta=-\pi/2}^{\theta=\pi/2} dz = 2\int_{-1}^{2} (1+2z^2)\,dz = 2\left(z + \frac{2}{3}z^3\right)\Big|_{z=-1}^{z=2} = 18 \end{aligned}$$

4. Use geometric reasoning to find $I = \iint_{\mathcal{S}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}}$ "by inspection" in the three situations below. Briefly explain your answers. (In all parts, a and b are positive constants.)

(a) $\vec{\mathbf{F}}(x, y, z) = x\vec{\imath} + y\vec{\jmath} + z\vec{\mathbf{k}}$, and $\mathcal{S}$ is the surface consisting of three squares with one corner at the origin and positive sides facing the first octant. The squares have sides $b\vec{\imath}$ and $b\vec{\jmath}$, $b\vec{\jmath}$ and $b\vec{\mathbf{k}}$, and $b\vec{\imath}$ and $b\vec{\mathbf{k}}$, respectively.

(b) $\vec{\mathbf{F}}(x, y, z) = (x\vec{\imath} + y\vec{\jmath}) \ln(x^2 + y^2)$, and $\mathcal{S}$ is the surface of the cylinder (including the top and bottom) where $x^2 + y^2 \le a^2$ and $0 \le z \le b$.

(c) $\vec{\mathbf{F}}(x, y, z) = (x\vec{\imath} + y\vec{\jmath} + z\vec{\mathbf{k}})e^{-(x^2+y^2+z^2)}$, and $\mathcal{S}$ is the spherical surface $x^2 + y^2 + z^2 = a^2$.

(a) The square with sides $b\vec{\imath}$ and $b\vec{\jmath}$ has normal $\hat{\mathbf{N}} = \vec{\mathbf{k}}$ and lies in the plane where $z = 0$. Thus $\vec{\mathbf{F}} \bullet \hat{\mathbf{N}} = z = 0$ on this part of the surface. The same thing happens on the other two squares, so we have

$$\iint_{\mathcal{S}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = \iint_{\mathcal{S}} \vec{\mathbf{F}} \bullet \hat{\mathbf{N}}\, dS = \iint_{\mathcal{S}} 0\, dS = 0.$$

(b) On the flat top of the cylinder, the outward normal is $\hat{\mathbf{N}} = \vec{\mathbf{k}}$, and we have $\vec{\mathbf{F}} \bullet \hat{\mathbf{N}} = 0$. Similarly on the flat bottom. On the sides, the outward unit normal at position (x, y, z) is clearly $\hat{\mathbf{N}} = \left(\frac{x}{a}, \frac{y}{a}, 0\right)$, so we have

$$\vec{\mathbf{F}} \bullet \hat{\mathbf{N}} = \left(x\left(\frac{x}{a}\right) + y\left(\frac{y}{a}\right)\right) \ln(x^2 + y^2) = a\ln(a^2) = 2a\ln(a).$$

It follows that

$$\iint_{\mathcal{S}} \vec{\mathbf{F}} \bullet \hat{\mathbf{N}}\, dS = 2a\ln(a)\ \text{area(curved side)} = 2a\ln(a)\,[2\pi ab] = 4\pi a^2 b\ln(a).$$

(c) On the surface of the sphere, the outward unit normal at $\vec{\mathbf{r}}$ is $\hat{\mathbf{N}} = \vec{\mathbf{r}}/|\vec{\mathbf{r}}| = \vec{\mathbf{r}}/a$. Hence

$$\vec{\mathbf{F}} \bullet \hat{\mathbf{N}} = \vec{\mathbf{r}}e^{-a^2} \bullet \frac{\vec{\mathbf{r}}}{a} = ae^{-a^2}.$$

Thus

$$\iint_{\mathcal{S}} \vec{\mathbf{F}} \bullet \hat{\mathbf{N}}\, dS = ae^{-a^2} \iint_{\mathcal{S}} dS = ae^{-a^2}\left[4\pi a^2\right] = 4\pi a^3 e^{-a^2}.$$

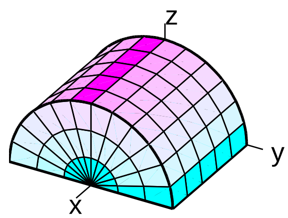
5. Let $\mathcal{S}$ be the boundary surface for the solid given by $0 \le z \le \sqrt{4-y^2}$ and $0 \le x \le \frac{\pi}{2}$.

   (a) Find the outward unit normal vector field $\hat{\mathbf{N}}$ on each of the four sides of $\mathcal{S}$.

   (b) Find the total outward flux of $\vec{\mathbf{F}} = 4\sin(x)\,\vec{\mathbf{i}} + z^3\,\vec{\mathbf{j}} + yz^2\,\vec{\mathbf{k}}$ through $\mathcal{S}$.

   Do the calculations directly—don't use the Divergence Theorem. [Hint: Flux integrals for three of the four sides can be calculated geometrically.]

(a) Here is a sketch of S:



On the front, where $x = \pi/2$, the outward unit normal is $\hat{\mathbf{N}} = \vec{\mathbf{i}}$.
On the back, where $x = 0$, the outward unit normal is $\hat{\mathbf{N}} = -\vec{\mathbf{i}}$.
On the bottom, where $z = 0$, the outward unit normal is $\hat{\mathbf{N}} = -\vec{\mathbf{k}}$.
On the top, where $y^2 + z^2 = 4$, an outward normal is $\vec{\mathbf{n}} = (0, y, z)$—either by geometric inspection or by finding a gradient. It follows that the outward *unit* normal is $\vec{\mathbf{n}}/|\vec{\mathbf{n}}| = (0, y/2, z/2)$.

(b) Recall $\vec{\mathbf{F}} = 4\sin x\,\vec{\mathbf{i}} + z^3\,\vec{\mathbf{j}} + yz^2\,\vec{\mathbf{k}}$.

On the front, where $x = \pi/2$ and $\hat{\mathbf{N}} = \vec{\mathbf{i}}$, $\vec{\mathbf{F}} \bullet \hat{\mathbf{N}} = 4\sin(\pi/2) = 4$, so

$$I_{\text{front}} = \iint_{\mathcal{S}_{\text{front}}} \vec{\mathbf{F}} \bullet \hat{\mathbf{N}}\,dS = 4\iint_{\mathcal{S}_{\text{front}}} dS = 4\ \text{area}(\mathcal{S}_{\text{front}}) = 8\pi.$$

On the back, where $x = 0$ and $\hat{\mathbf{N}} = -\vec{\mathbf{i}}$, $\vec{\mathbf{F}} \bullet \hat{\mathbf{N}} = 4\sin(0) = 0$, so

$$I_{\text{back}} = \iint_{\mathcal{S}_{\text{back}}} \vec{\mathbf{F}} \bullet \hat{\mathbf{N}}\,dS = \iint_{\mathcal{S}_{\text{back}}} 0\,dS = 0.$$

On the bottom, where $z = 0$ and $\hat{\mathbf{N}} = -\vec{\mathbf{k}}$, $\vec{\mathbf{F}} \bullet \hat{\mathbf{N}} = 0$, so

$$I_{\text{bottom}} = \iint_{\mathcal{S}_{\text{bottom}}} \vec{\mathbf{F}} \bullet \hat{\mathbf{N}}\,dS = \iint_{\mathcal{S}_{\text{bottom}}} 0\,dS = 0.$$

On the top, where $y^2 + z^2 = 4$ and $\hat{\mathbf{N}} = (0, y/2, z/2)$, $\vec{\mathbf{F}} \bullet \hat{\mathbf{N}} = \frac{1}{2}yz^3 + \frac{1}{2}yz^3 = yz^3$, so

$$I_{\text{top}} = \iint_{\mathcal{S}_{\text{top}}} \vec{\mathbf{F}} \bullet \hat{\mathbf{N}}\,dS = \iint_{\mathcal{S}_{\text{top}}} yz^3\,dS = \text{average}\left(yz^3\right)\ \text{area}(\mathcal{S}_{\text{top}}) = 0.$$

Here average$\left(yz^3\right) = 0$ because the surface $\mathcal{S}_{\text{top}}$ has reflection symmetry across the plane $y = 0$ and the integrand is an odd function of y. Of course the same value can be found by grinding calculation (provided it's done correctly).

Summary: The total outward flux of $\vec{\mathbf{F}}$ is the sum of four contributions, three of which are 0. The value is 8π.

6. Simplify the following expressions for smooth vector fields $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ and smooth scalar fields ϕ and ψ. [Hint: You may find Theorem 3 on pages 954–955 helpful.]

(a) $\nabla \bullet (\nabla\phi \times \nabla\psi)$

(b) $\nabla \bullet (\phi\vec{\mathbf{F}} + \vec{\mathbf{G}}) - (\nabla\phi) \bullet \vec{\mathbf{F}}$ for solenoidal $\vec{\mathbf{F}}$

(c) $\operatorname{div}\big(\vec{\mathbf{F}} \times (\vec{\mathbf{F}} + \vec{\mathbf{G}})\big)$ for conservative $\vec{\mathbf{G}}$

(a) Using Theorem 3(d) with $\vec{\mathbf{F}} = \nabla\phi$ and $\vec{\mathbf{G}} = \nabla\psi$, we get

$$\nabla \bullet (\nabla\phi \times \nabla\psi) = (\nabla \times (\nabla\phi)) \bullet (\nabla\psi) - (\nabla\phi) \bullet (\nabla \times (\nabla\psi))$$

However, by Theorem 3(h), $\nabla \times (\nabla\phi) = \vec{\mathbf{0}}$ and $\nabla \times (\nabla\psi) = \vec{\mathbf{0}}$, so

$$\nabla \bullet (\nabla\phi \times \nabla\psi) = \vec{\mathbf{0}} \bullet (\nabla\psi) - (\nabla\phi) \bullet \vec{\mathbf{0}} = 0 - 0 = 0$$

(b) We can use the linearity of the divergence operator followed by an application of Theorem 3(b) to write:

$$\begin{aligned}\nabla \bullet (\phi\vec{\mathbf{F}} + \vec{\mathbf{G}}) - (\nabla\phi) \bullet \vec{\mathbf{F}} &= \nabla \bullet (\phi\vec{\mathbf{F}}) + \nabla \bullet \vec{\mathbf{G}} - (\nabla\phi) \bullet \vec{\mathbf{F}} \\ &= (\nabla\phi) \bullet \vec{\mathbf{F}} + \phi(\nabla \bullet \vec{\mathbf{F}}) + \nabla \bullet \vec{\mathbf{G}} - (\nabla\phi) \bullet \vec{\mathbf{F}} \\ &= \phi(\nabla \bullet \vec{\mathbf{F}}) + \nabla \bullet \vec{\mathbf{G}}\end{aligned}$$

Because $\vec{\mathbf{F}}$ is solenoidal, we have $\nabla \bullet \vec{\mathbf{F}} = 0$, and so the final answer is $\nabla \bullet \vec{\mathbf{G}}$.

(c) By the linearity of the cross product and the divergence operator, we may write

$$\operatorname{div}\big(\vec{\mathbf{F}} \times (\vec{\mathbf{F}} + \vec{\mathbf{G}})\big) = \operatorname{div}\big(\vec{\mathbf{F}} \times \vec{\mathbf{F}} + \vec{\mathbf{F}} \times \vec{\mathbf{G}})\big) = \operatorname{div}\big(\vec{\mathbf{F}} \times \vec{\mathbf{F}}\big) + \operatorname{div}\big(\vec{\mathbf{F}} \times \vec{\mathbf{G}}\big)$$

At this point, you could either note that the cross product of any vector with itself is $\vec{\mathbf{0}}$, so we have $\vec{\mathbf{F}} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$ for any field $\vec{\mathbf{F}}$, or you could apply Theorem 3(d) to get

$$\operatorname{div}(\vec{\mathbf{F}} \times \vec{\mathbf{F}}) = (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{F}} - \vec{\mathbf{F}} \bullet (\nabla \times \vec{\mathbf{F}}) = 0$$

In any event, since

$$\operatorname{div}(\vec{\mathbf{F}} \times \vec{\mathbf{G}}) = (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{G}} - \vec{\mathbf{F}} \bullet (\nabla \times \vec{\mathbf{G}}) = (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{G}}$$

(with this last equality a consequence of $\vec{\mathbf{G}}$ conservative implying $\nabla \times \vec{\mathbf{G}} = \vec{\mathbf{0}}$). we have

$$\operatorname{div}\big(\vec{\mathbf{F}} \times (\vec{\mathbf{F}} + \vec{\mathbf{G}})\big) = \operatorname{div}(\vec{\mathbf{F}} \times \vec{\mathbf{G}}) = (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{G}}$$

7. A vector field $\vec{\mathbf{F}}$ is called a *curl field* if it can be expressed as $\vec{\mathbf{F}} = \text{curl}(\vec{\mathbf{G}})$ for some vector field $\vec{\mathbf{G}}$. In this case, $\vec{\mathbf{G}}$ is called a *vector potential* for $\vec{\mathbf{F}}$.

   (a) Explain why the following is true: if $\vec{\mathbf{G}}$ is a vector potential for a curl field $\vec{\mathbf{F}}$ and ϕ is a smooth scalar field, then $\vec{\mathbf{G}} + \nabla\phi$ is also a vector potential for $\vec{\mathbf{F}}$.

   Now, consider the vector field $\vec{\mathbf{F}} = \left\langle x^2e^{2y}, Aze^{2y}, (x-z)^2e^{2y}\right\rangle$ where A is a constant.

   (b) Only one choice for A makes $\vec{\mathbf{F}}$ a curl field. Find this value of A.

   (c) Using the value of A from part (b), find a vector potential for $\vec{\mathbf{F}}$ having special form $\vec{\mathbf{G}} = \langle G_1, 0, G_3\rangle$.

   (d) Repeat part (c), but find vector potentials with special forms $\langle 0, G_2, G_3\rangle$ and $\langle G_1, G_2, 0\rangle$. [Hint: Use the fact in part (a).]

(a) We have

$$\begin{aligned} \nabla \times (\vec{\mathbf{G}} + \nabla\phi) &= \nabla \times \vec{\mathbf{G}} + \nabla \times (\nabla\phi) && \text{(curl is linear)} \\ &= \nabla \times \vec{\mathbf{G}} && \text{(curl grad} = 0) \\ &= \vec{\mathbf{F}} && (\vec{\mathbf{G}} \text{ is a vector potential for } \vec{\mathbf{F}}) \end{aligned}$$

and this shows that $\vec{\mathbf{G}} + \nabla\phi$ is another vector potential for $\vec{\mathbf{F}}$.

(b) If $\vec{\mathbf{F}}$ is a curl field, it can be written $\vec{\mathbf{F}} = \text{curl}(\vec{\mathbf{G}})$ for some $\vec{\mathbf{G}}$. But, then we must have $\text{div}\,\vec{\mathbf{F}} = \text{div}(\text{curl}(\vec{\mathbf{G}})) = 0$ (because div curl $= 0$ always).

However, $\text{div}\,\vec{\mathbf{F}} = 0$ implies

$$\text{div}\,\vec{\mathbf{F}} = \frac{\partial}{\partial x}x^2e^{2y} + \frac{\partial}{\partial y}Aze^{2y} + \frac{\partial}{\partial z}(x-z)^2e^{2y} = 2xe^{2y} + 2Aze^{2y} - 2(x-z)e^{2y} = 0$$

and so $A = -1$.

(c) If $\vec{\mathbf{F}} = \text{curl}\,\langle G_1, 0, G_3\rangle$, then

$$\vec{\mathbf{F}} = \nabla \times \vec{\mathbf{G}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ G_1 & 0 & G_3 \end{vmatrix} = \left\langle \frac{\partial}{\partial y}G_3, \frac{\partial}{\partial z}G_1 - \frac{\partial}{\partial x}G_3, -\frac{\partial}{\partial y}G_1 \right\rangle$$

However, for $A = -1$, we have

$$\vec{\mathbf{F}} = \left\langle x^2e^{2y}, -ze^{2y}, (x-z)^2e^{2y}\right\rangle$$

and combining these two expressions for $\vec{\mathbf{F}}$ gives the following system of equations which we will denote by $(*)$:

$$(*) \begin{cases} \dfrac{\partial}{\partial y}G_3 = x^2e^{2y} \\ \dfrac{\partial}{\partial z}G_1 - \dfrac{\partial}{\partial x}G_3 = -ze^{2y} \\ \dfrac{\partial}{\partial y}G_1 = -(x-z)^2e^{2y} \end{cases}$$

Integrating the first and third equation of $(*)$ each with respect to y gives

$$\begin{aligned} G_3(x,y,z) &= \int x^2e^{2y}dy = \frac{1}{2}x^2e^{2y} + K_1(x,z) \\ G_1(x,y,z) &= \int -(x-z)^2e^{2y}dy = -\frac{1}{2}(x-z)^2e^{2y} + K_2(x,z) \end{aligned}$$

Substituting these into the second equation of $(*)$ gives

$$(x-z)e^{2y} + \frac{\partial}{\partial z}K_2(x,z) - xe^{2y} - \frac{\partial}{\partial x}K_1(x,z) = -ze^{2y}$$

which simplifies to

$$\frac{\partial}{\partial z}K_2(x,z) - \frac{\partial}{\partial x}K_1(x,z) = 0$$

Could we be so lucky that *any* functions $K_1(x,z)$ and $K_2(x,z)$ that satisfy this equation will work? Well, $K_1(x,z) = 0$ and $K_2(x,z) = 0$ satisfy this equation. Substituting them into the formulas for G_3 and G_1 above gives us:

$$\begin{aligned} G_3(x,y,z) &= \frac{1}{2}x^2e^{2y} \\ G_1(x,y,z) &= -\frac{1}{2}(x-z)^2e^{2y} \end{aligned}$$

Some quick partial differentiation confirms that these satisfy the system of equation $(*)$, giving the vector potential

$$\vec{\mathbf{G}} = \langle G_1, 0, G_3\rangle = \left\langle -\frac{1}{2}(x-z)^2e^{2y}, 0, \frac{1}{2}x^2e^{2y}\right\rangle$$

(d) In part (c), we found a vector potential for $\vec{\mathbf{F}}$, and part (a) tells us that we can add the gradient $\nabla\phi$ of any ϕ to our $\vec{\mathbf{G}}$ and *still* have a vector potential.

- To get a vector potential of the form $\langle 0, G_2, G_3\rangle$, we would like to find a ϕ so that

$$\langle 0, G_2, G_3\rangle = \left\langle -\frac{1}{2}(x-z)^2e^{2y}, 0, \frac{1}{2}x^2e^{2y}\right\rangle + \nabla\phi$$

or, in other words

$$\nabla\phi = \left\langle \frac{1}{2}(x-z)^2e^{2y}, G_2, G_3 - \frac{1}{2}x^2e^{2y}\right\rangle$$

for some as yet unknown G_2 and G_3. But, this implies

$$\frac{\partial}{\partial x}\phi = \frac{1}{2}(x-z)^2e^{2y}$$

which we integrate to show

$$\phi(x,y,z) = \int \frac{1}{2}(x-z)^2e^{2y}dx = \frac{1}{6}(x-z)^3e^{2y} + K_3(y,z)$$

Any choice of K_3 will work. Taking $K_3(y,z) = 0$, we have $\phi(x,y,z) = \frac{1}{6}(x-z)^3e^{2y}$ giving

$$\nabla\phi = \left\langle \frac{1}{2}(x-z)^2e^{2y}, \frac{1}{3}(x-z)^3e^{2y}, -\frac{1}{2}(x-z)^2e^{2y}\right\rangle$$

and so

$$\begin{aligned} \langle 0, G_2, G_3\rangle &= \left\langle -\frac{1}{2}(x-z)^2e^{2y}, 0, \frac{1}{2}x^2e^{2y}\right\rangle + \left\langle \frac{1}{2}(x-z)^2e^{2y}, \frac{1}{3}(x-z)^3e^{2y}, -\frac{1}{2}(x-z)^2e^{2y}\right\rangle \\ &= \left\langle 0, \frac{1}{3}(x-z)^3e^{2y}, \frac{1}{2}\left(x^2 - (x-z)^2\right)e^{2y}\right\rangle \end{aligned}$$

- To get a vector potential of the form $\langle G_1, G_2, 0\rangle$, we want a ϕ so that

$$\langle G_1, G_2, 0\rangle = \left\langle -\frac{1}{2}(x-z)^2e^{2y}, 0, \frac{1}{2}x^2e^{2y}\right\rangle + \nabla\phi$$

or

$$\nabla\phi = \left\langle G_1 + \frac{1}{2}(x-z)^2e^{2y}, G_2, -\frac{1}{2}x^2e^{2y}\right\rangle$$

for some as yet unknown G_1 and G_2. But, this implies

$$\frac{\partial}{\partial z}\phi = -\frac{1}{2}x^2e^{2y}$$

which we integrate to show

$$\phi(x,y,z) = \int -\frac{1}{2}x^2 e^{2y} dz = -\frac{1}{2}x^2 e^{2y} z + K_4(x,y)$$

Any choice of K_4 will work. Taking $K_4(x,y) = 0$, we have $\phi(x,y,z) = -\frac{1}{2}x^2 e^{2y} z$ giving

$$\nabla\phi = \left\langle -xe^{2y}z, -x^2e^{2y}z, -\frac{1}{2}x^2e^{2y} \right\rangle$$

and so

$$\begin{aligned}
\langle G_1, G_2, 0\rangle &= \left\langle -\frac{1}{2}(x-z)^2 e^{2y}, 0, \frac{1}{2}x^2e^{2y} \right\rangle + \left\langle -xe^{2y}z, -x^2e^{2y}z, -\frac{1}{2}x^2e^{2y} \right\rangle \\
&= \left\langle -\left(\frac{1}{2}(x-z)^2 + xz\right) e^{2y}, -x^2e^{2y}z, 0 \right\rangle
\end{aligned}$$