Math 263 HW07 Solutions

1. Compute the volumes of the following regions.

(a) The "ice-cream cone" region which is bounded above by the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ and below by the cone $z = \sqrt{x^2 + y^2}$.

Solution. In spherical coordinates,

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^a \rho^2 \sin\phi \, d\rho d\phi d\theta = 2\pi \int_0^{\pi/4} \sin\phi \rho^3/3\big]_0^a d\phi \\ &= 2\pi \left(-\cos\phi\right)\big]_0^{\pi/4} a^3/3 = \frac{2\pi a^3}{3}(-\cos\pi/4 + \cos 0) = \frac{\pi a^3(2-\sqrt{2})}{3} \end{aligned}$$

or in cylindrical coordinates,

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^{a/\sqrt{2}}\int_r^{\sqrt{a^2-r^2}} r dz dr d\theta = 2\pi \int_0^{a/\sqrt{2}} (r\sqrt{a^2-r^2} - r^2) dr \\ &= 2\pi \left[\frac{-(a^2-r^2)^{3/2} - r^3}{3}\right]_0^{a/\sqrt{2}} = 2\pi \frac{-(a^2-a^2/2)^{3/2} + (a^2-0^2)^{3/2} - (a/\sqrt{2})^3 + (0)^3}{3} \\ &= 2\pi\left(\frac{a^3 - 2a^3/2\sqrt{2}}{3}\right) = \frac{\pi a^3(2-\sqrt{2})}{3}. \end{aligned}$$

(b) The region bounded by $z = x^2 + 3y^2$ and $z = 4 - y^2$.

Solution. The parabolic cylinder $z = 4 - y^2$ comprises the top of the surface (considered in terms of z) and the paraboloid $z = x^2 + 3y^2$ is the bottom surface in terms of z. To determine the region of the xy-plane which the region bounded by these two surfaces lies over, we intersect the two surfaces, in this case we can set them equal to each other. We see that $x^2 + 3y^2 = 4 - y^2$ if and only if $x^2 + 4y^2 = 4$ if and only if $(x/2)^2 + y^2 = 1$. We will set up and compute the integral of this volume in rectangular coordinates (using a table of integrals to compute the anti-derivative $\int (4-x^2)^{3/2} dx$).

$$\begin{aligned} V &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_{x^2+3y^2}^{4-y^2} dz dy dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} (4 - x^2 - 4y^2) dy dx \\ &= \int_{-2}^{2} \left[(4-x^2)y - (4/3)y^3\right]_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} dx = 2\int_{-2}^{2}\left(\frac{(4-x^2)^{3/2}}{2} - \frac{(4-x^2)^{3/2}}{6}\right) dx \\ &= \frac{2}{3}\int_{-2}^{2}(4-x^2)^{3/2} dx = \frac{2}{3}\left[\frac{x}{8}\left(5\cdot 2^2 - 2x^2\right)\sqrt{4-x^2} + \frac{3\cdot 2^4}{8}\sin^{-1}(x/2)\right)\Big]_{-2}^{2} \\ &= 4(\sin^{-1}(1) - \sin^{-1}(-1)) = 4\pi \end{aligned}$$

Another way to compute this integral would be to make a substitution $x = 2u$, so $dx = 2du$ and we would be integrate over a circle of radius 1 in (u, y), which we will call $\widetilde{R}$ whereas the ellipse will be called R. This makes everything much simpler (I swear it does). Lets see what happens.

$$V = \int\int_R \left(\int_{x^2+3y^2}^{4-y^2} dz\right) dA = \int\int_R (4 - x^2 - 4y^2)dxdy = \int\int_{\widetilde{R}} (4 - 4u^2 - 4y^2)2dudy$$
$$= \int_0^{2\pi}\int_0^1 (4 - 4r^2)2rdrd\theta = \int_0^{2\pi} d\theta \int_0^1 (8r - 8r^3)dr = 2\pi\left[4r^2 - 2r^4\right]_0^1 = 2\pi(4-2) = 4\pi$$

(c) The region inside the sphere $x^2 + y^2 + z^2 = 9$ and outside the cylinder $x^2 + y^2 = 4$.

Solution. A sphere of radius 3 has volume $V_S = 36\pi$. Let V_C denote the volume inside the given sphere and the given cylinder simultaneously. The the volume we want, $V = V_S - V_C$. Let's compute V_C using cylindrical coordinates.

$$V_C = \int_0^{2\pi}\int_0^2\int_{-(9-r^2)^{1/2}}^{(9-r^2)^{1/2}} rdzdrd\theta = \int_0^{2\pi}\int_0^2 (9-r^2)^{1/2}2rdrd\theta$$
$$= 2\pi\left[\frac{-2}{3}(9-r^2)^{3/2}\right]_0^2 = \frac{4\pi}{3}(9^{3/2} - 5^{3/2}) = 36\pi - \frac{4\pi 5^{3/2}}{3};$$

hence $V = V_S - V_C = \frac{4\pi 5^{3/2}}{3}$.

2. Let T be the solid bounded by $z = 2$ and $z = \frac{1}{2}\sqrt{x^2+y^2}$ and with $y \geq 0$. Furthermore, assume that T has constant density $\delta(x, y, z) = \alpha > 0$.

(a) Compute the center of mass, $(\overline{x}, \overline{y}, \overline{z})$, of T. Recall that

Solution. Note that the solid T is one-half of a right-circular cone of height, $h = 2$, and radius, $r = 4$. Therefore, the volume of T, $V = (1/6)\pi r^2 h = (1/6)\pi 4^2 2 = 16\pi/3$. Since mass $= M = \int\int\int_T \delta(x, y, z)dV$ and $\delta(x, y, z) = \alpha$ which is a constant, we have that $M = 16\alpha\pi/3$. Also note that the region is symmetric about the yz-plane, and hence $\overline{x} = 0$. So we must compute $\overline{y}$ and $\overline{z}$, which is most naturally set-up in cylindrical coordinates. (Note that $z = (1/2)\sqrt{x^2+y^2} = r/2$ in cylindrical coordinates.)

$$\overline{y} = \frac{1}{M}\int\int\int_T y\delta(x,y,z)dV = \frac{3}{16\pi\alpha}\int\int\int_T y\alpha dzdydx = \frac{3}{16\pi}\int_0^{\pi}\int_0^4\int_{r/2}^2 r^2\sin\theta dzdrd\theta$$
$$= \frac{3}{16\pi}\int_0^{\pi}\int_0^4 (2 - r/2)r^2\sin\theta drd\theta = \frac{3}{16\pi}\left[(-\cos\theta)\right]_0^{\pi}\left[\frac{2}{3}r^3 - \frac{1}{8}r^4\right]_0^4$$
$$= \frac{3}{16\pi}(2)(128/3 - 256/8) = \frac{6}{\pi}(8/3 - 16/8) = 16/\pi - 12/\pi = 4/\pi$$

and

$$\overline{z} = \frac{1}{M}\int\int\int_T z\delta(x,y,z)dV = \frac{3}{16\pi\alpha}\int\int\int_T z\alpha dzdydx = \frac{3}{16\pi}\int_0^{\pi}\int_0^4\int_{r/2}^2 zrdzdrd\theta$$
$$= \frac{3}{16\pi}\int_0^{\pi}\int_0^4\left[\frac{z^2}{2}\right]_{r/2}^2 rdrd\theta = \frac{3}{16}\int_0^4\left(2 - \frac{r^2}{8}\right)rdr = \frac{3}{16}\left[r^2 - \frac{r^4}{32}\right]_0^4 = \frac{3}{4^2}(4^2 - \frac{4^2}{2}) = 3/2$$

So we have $(\overline{x}, \overline{y}, \overline{z}) = (0, 4/\pi, 3/2)$.

(b) Verify the mass of T using SPHERICAL coordinates.

Solution. We just need to determine the limits of integration is spherical coordinates. We have $0 \leq \theta \leq \pi$, since we are integrating over half a cone. The angle ϕ is measured from the z-axis down to the surface of the cone, which we determined has $h = 2$ and $r = 4$. So $\phi = \tan^{-1}(r/h) = \tan^{-1}(2)$. Finally, $0 \le \rho \le (z = 2)$, but in spherical coordinates $z = \rho\cos\phi = 2$ or $0 \le \rho \le 2\sec\phi$.

$$M = \int\int\int_T \delta(x,y,z)dV = \alpha\int\int\int_T dV = \alpha\int_0^{\pi}\int_0^{\tan^{-1}(2)}\int_0^{2\sec\phi}\rho^2\sin\phi d\rho d\phi d\theta$$
$$= \frac{\pi\alpha}{3}\int_0^{\tan^{-1}(2)}[\rho^3]_0^{2\sec\phi}\sin\phi d\phi = \frac{8\pi\alpha}{3}\int_0^{\tan^{-1}(2)}\sec^3\phi\sin\phi d\phi = \frac{8\pi\alpha}{3}\int_0^{\tan^{-1}(2)}\sec^2\phi\tan\phi d\phi$$
$$= \frac{8\pi\alpha}{3}\left[\frac{\tan^2\phi}{2}\right]_0^{\tan^{-1}(2)} = \frac{4\pi\alpha}{3}(4-0) = 16\pi\alpha/3$$

3. Let C be the helix $\mathbf{r}(t) = \langle\cos t, \sin t, t\rangle$ for $0 \le t \le 2\pi$ and let $f(x,y,z) = x^2 + y^2 + z^2$.
(a) Compute the arc length, s, of C.
Solution.

$$s = \int_C ds = \int_0^{2\pi}\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt = \int_0^{2\pi}\sqrt{(-\sin t)^2 + (\cos t)^2 + 1}dt = 2\pi\sqrt{2}.$$

(b) Evaluate $\int_C f(x,y,z)ds$.
Solution.

$$\int_C f(x,y,z)ds = \int_C (x^2+y^2+z^2)ds = \int_0^{2\pi}(\cos^2 t + \sin^2 t + t^2)\sqrt{2}dt = \sqrt{2}\int_0^{2\pi}(1+t^2)dt$$
$$= \sqrt{2}\left[t + \frac{t^3}{3}\right]_0^{2\pi} = \sqrt{2}\left(2\pi + \frac{8\pi^3}{3}\right) = 2\pi\sqrt{2}\left(1 + \frac{4\pi^2}{3}\right).$$

(c) Evaluate $\frac{1}{s}\int_C f(x,y,z)ds$, i.e., the average value of $f(x,y,z)$ along C.
Solution. From (a) and (b), we see that $\frac{1}{s}\int_C f(x,y,z)ds = 1 + 4\pi^2/3$

4. Compute $\int_C f(x,y,z)ds$ for the following curves and functions.
(a) $C : \mathbf{r}(t) = \langle 30\cos^3 t, 30\sin^3 t\rangle$ for $0 \le t \le \pi/2$ and $f(x,y) = 1 + y/3$.
Solution. First, $ds = |\mathbf{r}'(t)|dt = \sqrt{(-90\cos^2 t\sin t)^2 + (90\sin^2 t\cos t)^2}dt = 90\cos t\sin t dt$. Now we are in a position to compute the line integral.

$$\int_C (1+y/3)ds = \int_0^{\pi/2}(1 + 10\sin^3 t)90\cos t\sin t dt = \int_0^{\pi/2}(90\sin t + 900\sin^4 t)\cos t dt$$
$$= \int_{u=0}^{1}(90u + 900u^4)du, \text{ where } u = \sin t$$
$$= [45u^2 + 180u^5]_0^1 = 225$$

(b) $C : \mathbf{r}(t) = \langle t^2/2, t^3/3\rangle$ for $0 \le t \le 1$ and $f(x,y) = x^2 + y^2$.
Solution. Again we start by computing $ds = |\mathbf{r}'(t)|dt = t\sqrt{1+t^2}dt$. Then

$$\begin{aligned}\int_C (x^2+y^2)ds &= \int_0^1 ((t^2/2)^2+(t^3/3)^2)t\sqrt{1+t^2}dt = \frac{1}{4}\int_0^1 t^4\sqrt{1+t^2}(tdt)+\frac{1}{9}\int_0^1 t^6\sqrt{1+t^2}(tdt)\\
&= \frac{1}{8}\int_{u=1}^2 (u-1)^2\sqrt{u}du + \frac{1}{18}\int_{u=1}^2 (u-1)^3\sqrt{u}du, \text{ where } u = 1+t^2\\
&= \frac{1}{8}\int_1^2 (u^{5/2}-2u^{3/2}+u^{1/2})du + \frac{1}{18}\int_1^2 (u^{7/2}-3u^{5/2}+3u^{3/2}-u^{1/2})du\\
&= \left[\frac{u^{7/2}}{28}-\frac{u^{5/2}}{10}+\frac{u^{3/2}}{12}+\frac{u^{9/2}}{81}-\frac{u^{7/2}}{21}+\frac{u^{5/2}}{15}-\frac{u^{3/2}}{27}\right]_1^2\\
&= \left[\frac{u^{9/2}}{81}-\frac{u^{7/2}}{84}-\frac{u^{5/2}}{30}+\frac{5u^{3/2}}{108}\right]_1^2\\
&= (2^{9/2}/81-2^{7/2}/84-2^{5/2}/30+5\cdot 2^{3/2}/108)-(1/81-1/84-1/30+5/108)\end{aligned}$$

(c) $C : \mathbf{r}(t) = \langle 1, 2, t^2\rangle$ for $0 \leq t \leq 1$ and $f(x,y,z) = e^{\sqrt{z}}$.

Solution.

$$\int_C e^{\sqrt{z}}ds = \int_0^1 e^t\sqrt{0^2+0^2+(2t)^2}dt = \int_0^1 2te^t dt = [2te^t - 2e^t]_0^1 = 2$$

Note that we had to integrate by parts to anti-differentiate $2te^t$. (You let $u = 2t$ and $dv = e^t$.)

5. Determine whether or not the following vector fields are conservative. In the cases where $\mathbf{F}$ is conservative, find a function φ such that $\mathbf{F}(x,y,z) = \nabla\varphi(x,y,z)$.

(a) $\mathbf{F} = (2xy+z^2)\mathbf{i} + (x^2+2yz)\mathbf{j} + (y^2+2xz)\mathbf{k}$.

Solution. We first test to determine whether or not $\mathbf{F}$ might be conservative. Letting $F_1 = 2xy + z^2$, $F_2 = x^2 + 2yz$, and $F_3 = y^2 + 2xy$ (as usual), it is easy to verify that $\partial F_1/\partial y = \partial F_2/\partial x$, $\partial F_1/\partial z = \partial F_3/\partial x$, and $\partial F_2/\partial z = \partial F_3/\partial y$. There are many ways to find a function $\varphi(x,y,z)$ such that $\nabla\varphi = \mathbf{F}$, which is what we need to find. Here is one method. We will take anit-derivatives of F_1 with respect to x, F_2 with respect to y, and F_3 with respect to z respectively and then compare the results.

$$\begin{aligned}\varphi(x,y,z) &= \int (2xy+z^2)dx = x^2y + xz^2 + C_1(y,z)\\
\varphi(x,y,z) &= \int (x^2+2yz)dy = x^2y + y^2z + C_2(x,z)\\
\varphi(x,y,z) &= \int (y^2+2xz)dz = y^2z + xz^2 + C_3(x,y)\end{aligned}$$

It is very important that $C_1(y,z)$ is function of y and z and not just a constant, since we are "undoing" a partial derivative where we considered y and z as constants (similarly for $C_2(x,z)$ and $C_3(x,y)$). If we examine the three versions of $\varphi(x,y,z)$ we see that each version has at least one term in common. Therefore, we might try $\varphi(x,y,z) = x^2y + y^2z + xz^2$, which turns out to work in this case.

(b) $\mathbf{F} = (\ln(xy))\mathbf{i} + (\frac{x}{y})\mathbf{j} + (y)\mathbf{k}$.

Solution. Note that $\mathbf{F}$ is only defined for $x, y > 0$ or $x, y < 0$ and $F_1 = \ln(xy)$, $F_2 = x/y$, and $F_3 = y$ have continuous partials in these regions of the plane. Further, if $\mathbf{F} = \nabla\varphi$, and hence $\mathbf{F}$ is conservative, then the mixed second partials of φ must be equal. But since $\partial F_2/\partial z = 0$ and $\partial F_3/\partial y = 1$, no such φ could exist with $\nabla\varphi = (\ln(xy))\mathbf{i} + (\frac{x}{y})\mathbf{j} + (y)\mathbf{k}$.

(c) $\mathbf{F} = (e^x \cos y)\mathbf{i} + (-e^x \sin y)\mathbf{j} + (2z)\mathbf{k}$.

Solution. By inspection, it is easy to see that $\varphi(x, y, z) = z^2 + e^x \cos y$ is a potential function for $\mathbf{F}$. Otherwise, one could use a method similar to (a).

(d) $\mathbf{F} = (3x^2 y)\mathbf{i} + (4xy^2)\mathbf{j}$.

Solution. $\mathbf{F}$ is not conservative because $\partial F_1/\partial y = 3x^2 \neq 4y^2 = \partial F_2/\partial x$.

6. Let C_1 be the piece of the parabola $y = x^2$ from $P = (0, 0)$ to $Q = (1, 1)$ and let C_2 be the straight line from P to Q.

(a) Compute $\int_{C_1} y^2 dx + (x - y)dy$.

Solution. First we need to parameterize C_1. Let $\mathbf{r_1}(t) = \langle t, t^2 \rangle$, for $0 \leq t \leq 1$. Then $\mathbf{r_1'}(t) = \langle 1, 2t \rangle$. So

$$\int_{C_1} y^2 dx + (x - y)dy = \int_0^1 (t^2)^2 dt + (t - t^2) 2t dt = \left[\frac{t^5}{5} + \frac{2t^3}{3} + \frac{-t^4}{2} \right]_0^1 = \frac{1}{5} + \frac{2}{3} - \frac{1}{2} = \frac{11}{30}$$

(b) Compute $\int_{C_2} y^2 dx + (x - y)dy$.

Solution. We parameterize C_2 by $\mathbf{r_2}(t) = \langle t, t \rangle$, for $0 \leq t \leq 1$. Then $\mathbf{r_2'}(t) = \langle 1, 1 \rangle$. So

$$\int_{C_2} y^2 dx + (x - y)dy = \int_0^1 t^2 dt + (t - t)dt = \int_0^1 t^2 dt = [t^3/3]_0^1 = 1/3$$

(c) Based on your answers to (a) and (b), is the vector field $\mathbf{F}(x, y) = \langle y^2, x - y \rangle$ conservative? Why or why not?

Solution. If $\mathbf{F}(x, y) = \langle y^2, x - y \rangle$ were conservative, then the integral from A to B would have been independent of path. But note that the answers to (a) and (b) differ, therefore $\mathbf{F}$ is not conservative.

(d) Let $C = C_1 - C_2$ denote the end-to-end concatenation of C_1 with its given orientation and C_2 with the reverse orientation. Compute $\int_C xy^2 dx + x^2 y dy$.

Solution. Note that the vector field $\mathbf{F} = \langle xy^2, x^2 y \rangle$ is conservative, since $\partial(xy^2)/\partial y = 2xy = \partial(x^2 y)/\partial x$. Since C is a closed path and $\mathbf{F}$ is conservative, we know that

$$\int_C xy^2 dx + x^2 y dy = \int_{(0,0)}^{(0,0)} xy^2 dx + x^2 y dy = \varphi(0, 0) - \varphi(0, 0) = 0.$$

(Clearly $\varphi(x, y) = (x^2 y^2)/2$ gives a potential function for $\mathbf{F}$. However it does not matter that we can find a potential function for $\mathbf{F} = \langle xy^2, x^2 y \rangle$, we only needed that $\mathbf{F}$ was conservative.)