M263(2004) Solutions—Assignment 6

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1. The first calculation relies on the substitution $u = y^2$, $du = 2y\,dy$.

$$I = \int_0^2 \int_0^y y^2 e^{xy}\,dx\,dy = \int_0^2 y^2 \left[\frac{e^{xy}}{y}\right]_{x=0}^{y} dy = \int_0^2 [ye^{y^2} - y]\,dy$$
$$= \int_0^2 ye^{y^2}\,dy - \int_0^2 y\,dy = \tfrac{1}{2}\left[e^{y^2} - y^2\right]_{y=0}^{2} = \tfrac{1}{2}[e^4 - 4] - \tfrac{1}{2}[1-0] = \frac{e^4 - 5}{2}.$$

The second is completely straightforward:

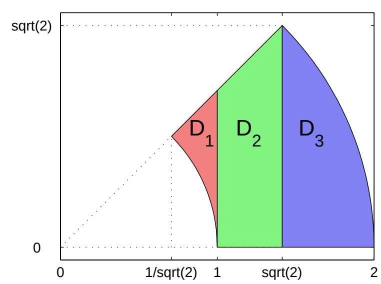
$$J = \int_0^\pi \int_{-x}^{x} \cos y\,dy\,dx = \int_0^\pi \left[\sin y\right]_{y=-x}^{x} dx$$
$$= \int_0^\pi 2\sin x\,dx = 2\left[-\cos x\right]_{x=0}^{\pi} = 4.$$

2. (a) Recognize $\quad I = \iint_D 1\,dA = \text{area(D)}, \quad$ where $\quad D = D_1 \cup D_2 \cup D_3.$

D_1: Projection *along* y (inner variable) *onto* x (outer variable) fills $1/\sqrt{2} \le x \le 1$; vertical filament at x runs from low $y = \sqrt{1-x^2}$ (a circular arc) up to high $y = x$ (a line). See sketch.

D_2: Projection *along* y *onto* x fills $1 \le x \le \sqrt{2}$; vertical filament at x runs from low $y = 0$ (a line) to high $y = x$ (a line). See sketch.

D_3: Projection *along* x *onto* y fills $0 \le y \le \sqrt{2}$; horizontal fibre at level y runs from low $x = \sqrt{2}$ (a line) to high $x = \sqrt{4-y^2}$ (a circular arc). See sketch.



(b) When $f(x,y) = K$, $I[f] = \iint_D f(x,y)\,dA = K \iint_D dA = K\,\text{Area}(D)$. So, by basic geometry,

$$I[f] = K \cdot \frac{1}{8}\left[\pi R^2 - \pi r^2\right]_{r=1,\,R=2} = \frac{3\pi}{8}K.$$

(c) In polar coordinates, region D has the simple description

$$0 \le \theta \le \frac{\pi}{4}, \qquad 1 \le r \le 2.$$

Also, $\sqrt{x^2+y^2} = r$. So when $f(x,y) = \sqrt{x^2+y^2}$,

$$I[f] = \iint_D \sqrt{x^2+y^2}\,dA = \int_{\theta=0}^{\pi/4} \int_{r=1}^{2} (r)\,r\,dr\,d\theta = \frac{\pi}{4}\left[\frac{r^3}{3}\right]_{r=1}^{2} = \frac{7\pi}{12}.$$

3. For I, the domain lies in the strip $0 \le y \le 1$, to the right of $x = y$ and the left of $x = 1$. Equivalently, it fills the part of the strip $0 \le x \le 1$ above $y = 0$ and below $y = x$. This gives

$$I = \int_{x=0}^{1} \int_{y=0}^{x} e^{-x^2}\, dy\, dx = \int_{x=0}^{1} e^{-x^2} \int_{y=0}^{x} dy\, dx = \int_{x=0}^{1} e^{-x^2} x\, dx$$

$$= \int_{u=0}^{1} e^{-u} \frac{du}{2} = -\frac{1}{2} \left[e^{-u}\right]_{u=0}^{1} = \tfrac{1}{2}\left(1 - e^{-1}\right).$$

For J, the domain lies in the strip $0 \le y \le \pi/2$, to the right of $x = y$ and to the left of $x = \pi/2$. That's another triangle. It can be re-described as the part of the strip $0 \le x \le \pi/2$ with left edge $y = 0$ and right edge $y = x$. Hence

$$J = \int_{x=0}^{\pi/2} \frac{\sin x}{x} \left(\int_{y=0}^{x} dy \right) dx = \int_{x=0}^{\pi/2} \left(\frac{\sin x}{x} \right) x\, dx = \int_{x=0}^{\pi/2} \sin x\, dx = 1.$$

For K, it's a similar story: the domain lies in the vertical strip $0 \le x \le 1$, above $y = x$ and below $y = 1$. Equivalently, it's the part of the horizontal strip $0 \le y \le 1$ above $y = x$ and below $y = 1$. Hence

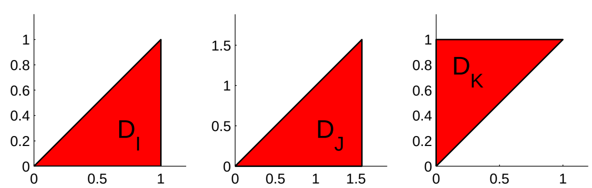
$$K = \int_{y=0}^{1} \int_{x=0}^{y} \frac{y^p}{x^2 + y^2}\, dx\, dy = \int_{y=0}^{1} y^{p-2} \left[\int_{x=0}^{y} \frac{1}{(x/y)^2 + 1}\, dx \right] dy$$

Let $u(x) = x/y$, $du = dx/y$ in the inner integral: the result is

$$K = \int_{y=0}^{1} y^{p-2} \left[\int_{u=0}^{1} \frac{1}{u^2 + 1} (y\, du) \right] dy = \int_{y=0}^{1} y^{p-1} \left[\tan^{-1}(u) \right]_{u=0}^{1} dy$$

$$= \frac{\pi}{4} \int_{y=0}^{1} y^{p-1}\, dy = \frac{\pi}{4p}.$$

The three domains described above are shown here. (Note that $\pi/2 \approx 1.57$.)



4. Call the given triangle T: it lies in the strip $0 \le y \le \pi^{1/4}$, to the right of $x = 0$ and left of $x = y$. So the volume we want is

$$V = \iint_T z\, dA = \int_{y=0}^{\pi^{1/4}} \int_{x=0}^{y} x^2 \sin(y^4)\, dx\, dy = \int_{y=0}^{\pi^{1/4}} \sin(y^4) \left[\frac{x^3}{3} \right]_{x=0}^{y} dy$$

$$= \frac{1}{3} \int_{y=0}^{\pi^{1/4}} \sin(y^4) y^3\, dy = \frac{1}{12} \int_{u=0}^{\pi} \sin(u)\, du = \frac{1}{6}.$$

5. Projecting region D along the y-direction onto the x-axis gives the iterated integrals below:

$$\text{Area}(D) = \iint_D 1\, dA = \int_{x=0}^{\infty} \int_{y=0}^{e^{-sx}} dy\, dx = \int_{x=0}^{\infty} e^{-sx}\, dx = \left[\frac{e^{-sx}}{(-s)} \right]_{x=0}^{x\to\infty} = \frac{1}{s},$$

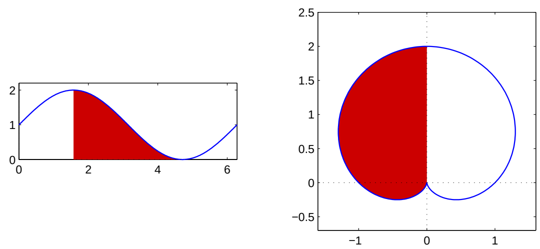
$$\iint_D x\, dA = \int_{x=0}^{\infty} \int_{y=0}^{e^{-sx}} x\, dy\, dx = \int_{x=0}^{\infty} x e^{-sx}\, dx = \left[x\frac{e^{-sx}}{(-s)} - \frac{e^{-sx}}{(-s)^2} \right]_{x=0}^{x\to\infty} = \frac{1}{s^2}$$

$$\iint_D y\, dA = \int_{x=0}^{\infty} \int_{y=0}^{e^{-sx}} y\, dy\, dx = \frac{1}{2} \int_{x=0}^{\infty} e^{-2sx}\, dx = \frac{1}{2} \left[\frac{e^{-2sx}}{(-2s)} \right]_{x=0}^{x\to\infty} = \frac{1}{4s}.$$

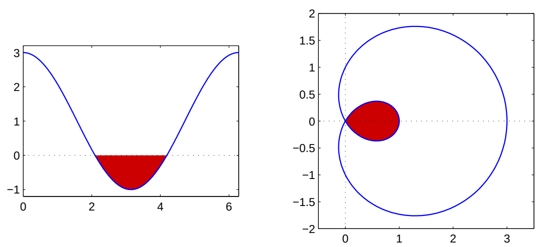
In view of the given definitions, we deduce that $\qquad \overline{x} = \dfrac{1/s^2}{1/s} = \dfrac{1}{s}, \qquad \overline{y} = \dfrac{1/(4s)}{1/s} = \dfrac{1}{4}.$

6. In the plots below, corresponding regions are shaded to emphasize their relationship.

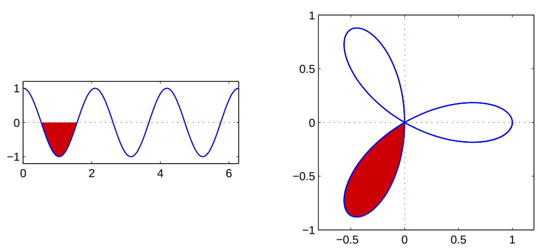
(i) $r = 1 + \sin\theta$: This is a cardioid.



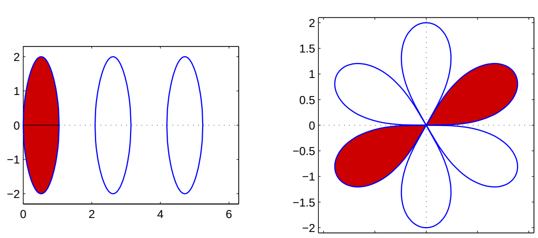
(ii) $r = 1 + 2\cos\theta$: Here a small interval of angles produces negative r-values, and these cause an inner loop in the polar plot.



(iii) $r = \cos 3\theta$: This is a three-leaved rose, traced twice in every interval of length 2π.



(iv) $r^2 = 4\sin 3\theta$: Here we obtain a real r-value only when the polar angle obeys $\sin 3\theta \geq 0$. But there are no sign restrictions on r, so we get $r = \pm 2\sqrt{\sin 3\theta}$ for all θ of this form.



7. The perimeter of the disk meets the vertical line $x = 1$ when $1 + y^2 = 2$, i.e., $y = \pm 1$. So in polar coordinates the disk segment is characterized by these inequalities:

$$S: \qquad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}, \quad \frac{1}{\cos\theta} \leq r \leq \sqrt{2}.$$

Consequently
$$\begin{aligned}\iint_S x\,dA &= \int_{\theta=-\pi/4}^{\pi/4}\int_{r=1/\cos\theta}^{\sqrt{2}} [r\cos\theta]\; r\,dr\,d\theta\\
&= \int_{\theta=-\pi/4}^{\pi/4} \cos\theta\left[\frac{r^3}{3}\right]_{r=1/\cos\theta}^{\sqrt{2}} d\theta\\
&= \frac{1}{3}\int_{\theta=-\pi/4}^{\pi/4}\left(2\sqrt{2}\cos\theta - \frac{1}{\cos^2\theta}\right) d\theta\\
&= \frac{1}{3}\left[2\sqrt{2}\sin\theta - \tan\theta\right]_{\theta=-\pi/4}^{\pi/4} = \frac{2}{3}.\end{aligned}$$

8. There is plenty of symmetry here. If the square is oriented so its diagonals lie on the lines $y = \pm x$, then these lines also divide the disk into regions where the nearest side of the square is easy to predict. The average distance over the whole disk will equal the average distance for points in the top wedge only. The polar inequalities characterizing this wedge (name it "D") are

$$D: \qquad \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}, \quad 0 \le r \le 1.$$

The distance from a point (x, y) in this wedge to the line $y = 1$ is simply $f(x, y) = 1 - y$: the average of this function over D is approximately 0.3998, because

$$\begin{aligned}\overline{f} = \frac{\iint_D f\,dA}{\iint_D 1\,dA} &= \frac{1}{\pi/4}\int_{\theta=\pi/4}^{3\pi/4}\int_{r=0}^{1} [1 - r\sin\theta]\, r\,dr\,d\theta\\
&= \frac{4}{\pi}\int_{\theta=\pi/4}^{3\pi/4}\left[\frac{1}{2} - \frac{1}{3}\sin\theta\right] d\theta = \frac{4}{\pi}\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{3}\sqrt{2}\right] = 1 - \frac{4\sqrt{2}}{3\pi}.\end{aligned}$$

9. The paraboloid meets the xy-plane where $z = 0$, i.e., where $x^2 + y^2 = 1$. Hence the desired volume lies above a sector in the xy-plane, whose straight sides lie along rays through the origin associated with angles $-\pi/4$ and $\pi/3$. In polar coordinates, this wedge can be described by

$$D: \qquad -\frac{\pi}{4} \le \theta \le \frac{\pi}{3}, \qquad 0 \le r \le 1.$$

The desired volume is

$$\begin{aligned}V = \iint_D (1 - x^2 - y^2)\,dA &= \int_{\theta=-\pi/4}^{\pi/3}\int_{r=0}^{1} (1 - r^2) r\,dr\,d\theta\\
&= \frac{7\pi}{12}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{7\pi}{48}.\end{aligned}$$

10. (a) Here $I = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$ is an iteration of the triple integral

$$I = \iiint_R z\sqrt{x^2+y^2+z^2}\,dV,$$

where R is the top half of a solid sphere with radius 3 centred at the origin. In spherical coordinates,

$$\begin{aligned} I &= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{3} [\rho\cos\phi]\sqrt{\rho^2}\,\rho^2\,\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \left(\int_{\phi=0}^{\pi/2} \sin\phi\cos\phi\,d\phi\right)\left(\int_{\theta=0}^{2\pi} d\theta\right)\int_{\rho=0}^{3} \rho^4\,d\rho \\ &= \left[\tfrac{1}{2}\sin^2\phi\right]_{\phi=0}^{\pi/2}(2\pi)\left(\frac{3^5}{5}\right) = \frac{243\pi}{5}. \end{aligned}$$

(b) Here $J = \displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} \left(x^2+y^2+z^2\right)\,dz\,dx\,dy$ is an iteration of the triple integral

$$J = \iiint_R \left(x^2+y^2+z^2\right)\,dV,$$

where R is a solid region in the first octant. The solid lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 18$. To check this, note that the cone meets the sphere at the height where $z^2+z^2 = 18$, i.e., $z = 3$, and the ring in which they intersect obeys $x^2+y^2 = 9$, as reflected in the limits of the outer double integral. The vertex angle of the cone (between $\mathbf{k}$ and a line on the slanted surface) is $\pi/4$, so in spherical coordinates

$$\begin{aligned} J &= \int_{\phi=0}^{\pi/4} \int_{\theta=0}^{\pi/2} \int_{\rho=0}^{\sqrt{18}} \left(\rho^2\right)\,\rho^2\,\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \left(\int_{\phi=0}^{\pi/4} \sin\phi\,d\phi\right)\left(\int_{\theta=0}^{\pi/2} d\theta\right)\int_{\rho=0}^{3\sqrt{2}} \rho^4\,d\rho \\ &= \Big[-\cos\phi\Big]_{\phi=0}^{\pi/4}\left(\frac{\pi}{2}\right)\left(\frac{(3\sqrt{2})^5}{5}\right) = \frac{486\pi}{5}(\sqrt{2}-1). \end{aligned}$$

11. Use spherical coordinates, where $\rho = |\mathbf{x}|$ and $dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$:

$$\begin{aligned} Q &= \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{\infty} \rho e^{-\rho^2}\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \left(\int_{\phi=0}^{\pi} \sin\phi\,d\phi\right)\left(\int_{\theta=0}^{2\pi} d\theta\right)\int_{\rho=0}^{\infty} \rho^2 e^{-\rho^2}\,\rho\,d\rho \\ &= 4\pi\int_{u=0}^{\infty} ue^{-u}\left(\frac{du}{2}\right) \qquad (\text{sub } u=\rho^2,\ du = 2\rho\,d\rho) \\ &= 2\pi\left[-ue^{-u}\Big|_{u=0}^{\infty} - \int_0^{\infty}\left(-e^{-u}\right)\,du\right] = 2\pi. \end{aligned}$$

The final step involved integration by parts and two famous limits: both $e^{-u} \to 0$ and $ue^{-u} \to 0$ as $u \to +\infty$.