## MATH 263 ASSIGNMENT 5 SOLUTIONS

1) If $t_{0}$ is a local minimum or maximum of the smooth function $f(t)$ of one variable ( $t$ runs over all real numbers) then $f^{\prime}\left(t_{0}\right)=0$. Derive an analogous necessary condition for $\vec{x}_{0}$ to be a local minimum or maximium of the smooth function $g(\vec{x})$ restricted to points on the line $\vec{x}=\vec{a}+t \vec{d}$. The test should involve the gradient of $g(\vec{x})$.
Solution. Define $f(t)=g(\vec{a}+t \vec{d})$ and determine $t_{0}$ by $\vec{x}_{0}=\vec{a}+t_{0} \vec{d}$. Then $f^{\prime}(t)=\vec{\nabla} g(\vec{a}+t \vec{d}) \cdot \vec{d}$. Then $\vec{x}_{0}$ is a local max or min of the restriction of $g$ to the specified line if and only if $t_{0}$ is a local max or min of $f(t)$. If so, $f^{\prime}\left(t_{0}\right)$ necessarily vanishes. So if $\vec{x}_{0}$ is a local max or min of the restriction of $g$ to the specified line, then $\vec{\nabla} g\left(x_{0}\right) \perp \vec{d}$ and $\vec{x}_{0}=\vec{a}+t_{0} \vec{d}$ for some $t_{0}$. The second condition is to ensure that $x_{0}$ lies on the line.
2) Find the maximum and minimum values of $f(x, y)=x y-x^{3} y^{2}$ when $(x, y)$ runs over the square $0 \leq x \leq 1,0 \leq y \leq 1$.

Solution.

$$
f(x, y)=x y-x^{3} y^{2} \quad f_{x}(x, y)=y-3 x^{2} y^{2} \quad f_{y}(x, y)=x-2 x^{3} y
$$

First, we find the critical points

$$
\begin{aligned}
& f_{x}=0 \\
& f_{y}=0
\end{aligned} \Longleftrightarrow y\left(1-3 x^{2} y\right)=0 \quad \Longleftrightarrow \quad y=0 \text { or } 3 x^{2} y=1 ~ 子 x\left(1-2 x^{2} y\right)=0 \quad \Longleftrightarrow \quad x=0 \text { or } 2 x^{2} y=1 ~ \$
$$

If $y=0$, we cannot have $2 x^{2} y=1$, so we must have $x=0$. If $3 x^{2} y=1$, we cannot have $x=0$, so we must have $2 x^{2} y=1$. Dividing gives $1=\frac{3 x^{2} y}{2 x^{2} y}=\frac{3}{2}$ which is impossible. So the only critical point in the square is $(0,0)$. There $f=0$.
Next, we look at the part of the boundary with $x=0$. There $f=0$.
Next, we look at the part of the boundary with $y=0$. There $f=0$.
Next, we look at the part of the boundary with $x=1$. There $f=y-y^{2}$. As $\frac{d}{d y}\left(y-y^{2}\right)=1-2 y$, the $\max$ and min of $y-y^{2}$ for $0 \leq y \leq 1$ must occur either at $y=0$, where $f=0$, or at $y=\frac{1}{2}$, where $f=\frac{1}{4}$, or at $y=1$, where $f=0$.
Next, we look at the part of the boundary with $y=1$. There $f=x-x^{3}$. As $\frac{d}{d x}\left(x-x^{3}\right)=1-3 x^{2}$, the max and min of $x-x^{3}$ for $0 \leq x \leq 1$ must occur either at $x=0$, where $f=0$, or at $x=\frac{1}{\sqrt{3}}$, where $f=\frac{2}{3 \sqrt{3}}$, or at $x=1$, where $f=0$.
All together, we have the following candidates for max and min

| point | $(0,0)$ | $(0,0 \leq y \leq 1)$ | $(0 \leq x \leq 1,0)$ | $(1,0)$ | $\left(1, \frac{1}{2}\right)$ | $(1,1)$ | $(0,1)$ | $\left(\frac{1}{\sqrt{3}}, 1\right)$ | $(1,1)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| value of $f$ | 0 | 0 | 0 | 0 | $\frac{1}{4}$ | 0 | 0 | $\frac{2}{3 \sqrt{3}}$ | 0 |

The largest and smallest values of $f$ in this table are $\min =0, \max =\frac{2}{3 \sqrt{3}} \approx 0.385$.
3) The temperature at all points in the disc $x^{2}+y^{2} \leq 1$ is given by $T(x, y)=(x+y) e^{-x^{2}-y^{2}}$. Find the maximum and minimum temperatures at points of the disc.
Solution.
$T(x, y)=(x+y) e^{-x^{2}-y^{2}} \quad T_{x}(x, y)=\left(1-2 x^{2}-2 x y\right) e^{-x^{2}-y^{2}} \quad T_{y}(x, y)=\left(1-2 x y-2 y^{2}\right) e^{-x^{2}-y^{2}}$

First, we find the critical points

$$
\begin{array}{lll}
T_{x}=0 & \Longleftrightarrow & 2 x(x+y)=1 \\
T_{y}=0 & \Longleftrightarrow & 2 y(x+y)=1
\end{array}
$$

As $x+y$ may not vanish, this forces $x=y$ and then $x=y= \pm \frac{1}{2}$. So the only critical points are $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2},-\frac{1}{2}\right)$.
On the boundary $x=\cos t$ and $y=\sin t$, so $T=(\cos t+\sin t) e^{-1}$. This is a periodic function and so takes its max and min at zeroes of $\frac{d T}{d t}=(-\sin t+\cos t) e^{-1}$. That is, when $\sin t=\cos t$, which forces $\sin t=\cos t= \pm \frac{1}{\sqrt{2}}$. All together, we have the following candidates for max and min

| point | $\left(\frac{1}{2}, \frac{1}{2}\right)$ | $\left(-\frac{1}{2},-\frac{1}{2}\right)$ | $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ | $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| value of $f$ | $\frac{1}{\sqrt{e}} \approx 0.61$ | $-\frac{1}{\sqrt{e}}$ | $\frac{\sqrt{2}}{e} \approx 0.52$ | $-\frac{\sqrt{2}}{e}$ |

The largest and smallest values of $T$ in this table are $\min =-\frac{1}{\sqrt{e}}, \max =\frac{1}{\sqrt{e}}$.
4) Find the high and low points of the surface $z=\sqrt{x^{2}+y^{2}}$ with $(x, y)$ varying over the square $|x| \leq 1$, $|y| \leq 1$. Discuss the values of $z_{x}, z_{y}$ there. Do not evaluate any derivatives in answering this question.
Solution. The surface is a cone. The minimum height is at $(0,0,0)$. The cone has a point there and the derivatives $z_{x}$ and $z_{y}$ do not exist. The maximum height is achieved when $(x, y)$ is as far as possible from $(0,0)$. The highest points are at $( \pm 1, \pm 1, \sqrt{2})$. There $z_{x}$ and $z_{y}$ exist but are not zero. These points would not be the highest points if it were not for the restriction $|x|,|y| \leq 1$.
5) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z)=x+y-z$ on the sphere $x^{2}+y^{2}+z^{2}=1$.
Solution. Define $L(x, y, z, \lambda)=x+y-z-\lambda\left(x^{2}+y^{2}+z^{2}-1\right)$. Then

$$
\begin{array}{ll}
0=L_{x}=1-2 \lambda x & \Longrightarrow x=\frac{1}{2 \lambda} \\
0=L_{y}=1-2 \lambda y & \Longrightarrow y=\frac{1}{2 \lambda} \\
0=L_{z}=-1-2 \lambda x & \Longrightarrow z=-\frac{1}{2 \lambda} \\
0=L_{\lambda}=x^{2}+y^{2}+z^{2}-1 & \Longrightarrow 3\left(\frac{1}{2 \lambda}\right)^{2}-1=0 \quad \Longrightarrow \lambda= \pm \frac{\sqrt{3}}{2}
\end{array}
$$

The critical points are $\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, where $f=-\sqrt{3}$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$, where $f=\sqrt{3}$. So, the max is $f=\sqrt{3}$ and the min is $f=-\sqrt{3}$.
6) Find $a, b$ and $c$ so that the volume $4 \pi a b c / 3$ of an ellipsoid $\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}+\frac{z^{2}}{c^{2}}=1$ passing through the point $(1,2,1)$ is as small as possible.
Solution. Define $L(a, b, c, \lambda)=\frac{4}{3} \pi a b c-\lambda\left(\frac{1}{a^{2}}+\frac{4}{b^{2}}+\frac{1}{c^{2}}-1\right)$. Then

$$
\begin{array}{ll}
0=L_{a}=\frac{4}{3} \pi b c+\frac{2 \lambda}{a^{3}} & \Longrightarrow \frac{3}{2 \pi} \lambda=-a^{3} b c \\
0=L_{b}=\frac{4}{3} \pi a c+\frac{8 \lambda}{b^{3}} & \Longrightarrow \frac{3}{2 \pi} \lambda=-\frac{1}{4} a b^{3} c \\
0=L_{c}=\frac{4}{3} \pi a b+\frac{2 \lambda}{c^{3}} & \Longrightarrow \frac{3}{2 \pi} \lambda=-a b c^{3} \\
0=L_{\lambda}=\frac{1}{a^{2}}+\frac{4}{b^{2}}+\frac{1}{c^{2}}-1 &
\end{array}
$$

The equations $-\frac{3}{2 \pi} \lambda=a^{3} b c=\frac{1}{4} a b^{3} c$ force $b=2 a$ (since we want $a, b, c>0$ ). The equations $-\frac{3}{2 \pi} \lambda=$ $a^{3} b c=a b c^{3}$ force $a=c$. Hence

$$
0=\frac{1}{a^{2}}+\frac{4}{b^{2}}+\frac{1}{c^{2}}-1=\frac{3}{a^{2}}-1 \Longrightarrow a=c=\sqrt{3}, b=2 \sqrt{3}
$$

7) Find the ends of the major and minor axes of the ellipse $3 x^{2}-2 x y+3 y^{2}=4$.

Solution. Let $(x, y)$ be a point on $3 x^{2}-2 x y+3 y^{2}=4$. This point is at the end of a major axis when it maximizes its distance from the centre, $(0,0)$ of the ellipse. It is at the end of a minor axis when it minimizes its distance from $(0,0)$. So we wish to maximize and minimize $x^{2}+y^{2}$ subject to $3 x^{2}-2 x y+3 y^{2}=4$. Define $L(x, y, \lambda)=x^{2}+y^{2}-\lambda\left(3 x^{2}-2 x y+3 y^{2}-4\right)$. Then

$$
\begin{array}{ll}
0=L_{x}=2 x-\lambda(6 x-2 y) & \Longrightarrow(1-3 \lambda) x+\lambda y=0 \\
0=L_{y}=2 y-\lambda(-2 x+6 y) & \Longrightarrow \lambda x+(1-3 \lambda) y=0  \tag{2}\\
0=L_{\lambda}=3 x^{2}-2 x y+3 y^{2}-4 &
\end{array}
$$

Note that $\lambda$ cannot be zero because if it is, (1) forces $x=0$ and (2) forces $y=0$. But $(0,0)$ is not on the ellipse and so is not an acceptable solution. As $\lambda \neq 0$, equation (1) gives $y=-\frac{1-3 \lambda}{\lambda} x$. Subbing this into equation (2) gives $\lambda x-\frac{(1-3 \lambda)^{2}}{\lambda} x=0$. To get a nonzero $(x, y)$ we need

$$
\lambda-\frac{(1-3 \lambda)^{2}}{\lambda}=0 \Longleftrightarrow \lambda^{2}-(1-3 \lambda)^{2}=0 \Longleftrightarrow[\lambda-(1-3 \lambda)][\lambda+(1-3 \lambda)]=0 \Longleftrightarrow[4 \lambda-1][1-2 \lambda]=0
$$

So $\lambda$ must be either $\frac{1}{2}$ or $\frac{1}{4}$. Subbing these into either (1) or (2) gives

$$
\begin{aligned}
& \lambda=\frac{1}{2} \Longrightarrow-\frac{1}{2} x+\frac{1}{2} y=0 \Longrightarrow x=y \Longrightarrow 3 x^{2}-2 x^{2}+3 x^{2}=4 \Longrightarrow x= \pm 1 \\
& \lambda=\frac{1}{4} \Longrightarrow \frac{1}{4} x+\frac{1}{4} y=0 \Longrightarrow x=-y \Longrightarrow 3 x^{2}+2 x^{2}+3 x^{2}=4 \Longrightarrow x= \pm \frac{1}{\sqrt{2}}
\end{aligned}
$$

The ends of the minor axes are $\pm\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$. The ends of the major axes are $\pm(1,1)$.
8) Find the triangle of largest area that can be inscribed in the circle $x^{2}+y^{2}=1$.

Solution. Inscribe the base of the triangle and choose a coordinate system in which the base is horizontal. Pick the vertex of the triangle. For a given base, the triangle has maximum height (and hence area) if the vertex is chosen to be at the "top" of the circle, as shown.


We are to mazimize $A=\frac{1}{2} b h$ subject to $(h-1)^{2}+\left(\frac{b}{2}\right)^{2}=1$. Define

$$
L(b, h, \lambda)=\frac{1}{2} b h-\lambda\left((h-1)^{2}+\left(\frac{b}{2}\right)^{2}-1\right)
$$

Then

$$
\begin{array}{lll}
0=L_{b}=\frac{1}{2} h-\frac{1}{2} \lambda b \\
0=L_{h}=\frac{1}{2} b-2 \lambda(h-1) \\
0=L_{\lambda}=-(h-1)^{2}-\left(\frac{b}{2}\right)^{2}+1 & \Rightarrow \lambda=\frac{b}{4(h-1)} & \Rightarrow \frac{b^{2}}{4}=h(h-1) \\
& & \\
& \Rightarrow(h-1)^{2}+h(h-1)=1 \\
& \Rightarrow 2 h^{2}-3 h=0
\end{array}
$$

So $h$ must be either 0 (which cannot give maximum area) or $h=\frac{3}{2}$ and $b=\sqrt{3}$. All three sides of the triangle have length $\sqrt{3}$, so the triangle is equilateral (surprise!).
9) The temperature gradient, at each point $(x, y)$ of the disk $x^{2}+y^{2} \leq 25$, is a strictly positive multiple of $(6+x, 8+y)$. Find the hottest point of the disk.
Solution. On the disk $-5 \leq x \leq 5$, so that $6+x$ can never be zero and the temperature gradient can never vanish. Thus the temperature has no critical points in the disk and the maximum and minimum temperatures must occur on the boundary $x^{2}+y^{2}=25$. By the method of Lagrange multipliers, the temperature gradient must be parallel to the normal vector to $x^{2}+y^{2}=25$, which is $(2 x, 2 y)$, at any extremal point on $x^{2}+y^{2}=25$. Hence at any extremal point there must be a $\lambda$ such that

$$
\begin{aligned}
6+x=2 \lambda x & \Longrightarrow x=\frac{6}{2 \lambda-1} \\
8+y=2 \lambda y & \Longrightarrow y=\frac{8}{2 \lambda-1} \\
x^{2}+y^{2}=25 & \Longrightarrow \frac{36}{(2 \lambda-1)^{2}}+\frac{64}{(2 \lambda-1)^{2}}=25 \Longrightarrow(2 \lambda-1)= \pm 2 \Longrightarrow(x, y)= \pm(3,4)
\end{aligned}
$$

Since $T_{x}=6+x>0$ and $T_{y}=8+y>0$, the temperature increases to the right and upward so that the temperature at $(3,4)$ must be higher than the temperature at $(-3,-4)$. The hottest point is $(3,4)$.
10) For each of the following, evaluate the given double integral without using iteration. Instead, interpret the integral as an area or some other physical quantity.
a) $\iint_{R} d x d y$ where $R$ is the rectangle $-1 \leq x \leq 3,-4 \leq y \leq 1$.
b) $\iint_{D}(x+3) d x d y$, where $D$ is the half disc $0 \leq y \leq \sqrt{4-x^{2}}$.
c) $\iint_{R}(x+y) d x d y$ where $R$ is the rectangle $0 \leq x \leq a, 0 \leq y \leq b$.
d) $\iint_{R} \sqrt{a^{2}-x^{2}-y^{2}} d x d y$ where $R$ is the region $x^{2}+y^{2} \leq a^{2}$.
e) $\iint_{R} \sqrt{b^{2}-y^{2}} d x d y$ where $R$ is the rectangle $0 \leq x \leq a, 0 \leq y \leq b$.

Solution. a) $\iint_{R} d x d y$ is the area of a rectangle with sides of lengths 4 and 5 . This area is $\iint_{R} d x d y=4 \times 5=20$.
b) $\iint_{D} x d x d y=0$ because $x$ is odd under reflection about the $y$-axis, while the domain of integration is symmetric about the $y$-axis. $\iint_{D} 3 d x d y$ is the three times the area of a half disc of radius 2 . So, $\iint_{D}(x+3) d x d y=3 \times \frac{1}{2} \times \pi 2^{2}=6 \pi$.
c) $\iint_{R} x d x d y / \iint_{R} d x d y$ is the average value of $x$ in the rectangle $R$, namely $\frac{a}{2}$. Similarly, $\iint_{R} y d x d y / \iint_{R} d x d y$ is the average value of $y$ in the rectangle $R$, namely $\frac{b}{2}$. $\iint_{R} d x d y$ is area of the rectangle $R$, namely $a b$. So, $\iint_{S}(x+y) d x d y=\frac{1}{2} a b(a+b)$.
d) $\iint_{R} \sqrt{a^{2}-x^{2}-y^{2}} d x d y$ is the volume of the region, $V$, with $0 \leq z \leq \sqrt{a^{2}-x^{2}-y^{2}}, x^{2}+y^{2} \leq a^{2}$. This is the top half of a sphere of radius $a$. So, $\iint_{R} \sqrt{a^{2}-x^{2}-y^{2}} d x d y=\frac{2}{3} \pi a^{3}$.
e) $\iint_{R} \sqrt{b^{2}-y^{2}} d x d y$ is the volume of the region, $V$, with $0 \leq z \leq \sqrt{b^{2}-y^{2}}, 0 \leq x \leq a, 0 \leq y \leq b$. $y^{2}+z^{2} \leq b^{2}$ is a cylinder of radius $b$ centered on the $x$ axis. $y^{2}+z^{2} \leq b^{2}, y \geq 0, z \geq 0$ is one quarter of this cylinder. It has cross-sectional area $\frac{1}{4} \pi b^{2}$. $V$ is the part of this quarter-cylinder with $0 \leq x \leq a$. It has length $a$ and cross-sectional area $\frac{1}{4} \pi b^{2}$. So, $\iint_{R} \sqrt{b^{2}-y^{2}} d x d y=\frac{1}{4} \pi a b^{2}$.

