Math 263 Assignment #4 Solutions

1. Find and classify the critical points of each of the following functions:

(a) $f(x,y,z) = x^2 + yz - x - 2y - z + 7$ (c) $f(x,y) = e^{-x^2-y^2}(1 - e^{x^2})$

(b) $f(x,y) = (x+y)^3 - (x-y)(x-5y)$ (d) $f(x,y) = 2\sin x \cos y$

(a) For $f(x,y,z) = x^2 + yz - x - 2y - z + 7$, setting $\nabla f(x,y,z) = \mathbf{0}$ gives the system of equations:

$$\begin{aligned} 0 &= f_1(x,y,z) = 2x - 1 \\ 0 &= f_2(x,y,z) = z - 2 \\ 0 &= f_3(x,y,z) = y - 1 \end{aligned}$$

which immediately gives $x = 1/2$, $z = 2$, and $y = 1$ and so the single critical point $(1/2, 1, 2)$.

To classify this critical point using the second derivative test, we first calculate the second-order partial derivatives:

$$\begin{array}{lll} f_{11}(x,y,z) = 2 & f_{12}(x,y,z) = 0 & f_{13} = 0 \\ & f_{22}(x,y,z) = 0 & f_{23} = 1 \\ & & f_{33} = 0 \end{array}$$

which give the Hessian

$$\mathcal{H}(x,y,z) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

for *all* points (x,y,z) and so in particular for our critical point. Using MATLAB, we find `eig(H)` gives eigenvalues -1, 1, and 2, and since there are positive and negative eigenvalues, the matrix is indefinite and so $(1/2, 1, 2)$ is a saddle point.

Alternatively, we could consider the change in f for small changes dx and dy away from the critical point:

$$\begin{aligned} \Delta f &= f(1/2 + dx, 1 + dy, 2 + dz) - f(1/2, 1, 2) \\ &= [(1/2 + dx)^2 + (1 + dy)(2 + dz) - (1/2 + dx) - 2(1 + dy) - (2 + dz) + 7] \\ &\qquad - [(1/2)^2 + (1)(2) - (1/2) - 2(1) - 2 + 7] \\ &= 1/4 + dx + dx^2 + 2 + 2dy + dz + dy\,dz - 1/2 - dx - 2 - 2dy - 2 - dz + 7 \\ &\qquad - 1/4 - 2 + 1/2 + 2 + 2 - 7 \\ &= dx^2 + dydz \end{aligned}$$

If we fixed $dx = 0$, then $dy > 0$ and $dz > 0$ would give $\Delta f > 0$ while $dy > 0$ and $dz < 0$ would give $\Delta f < 0$. Therefore, $(1/2, 1, 2)$ is a saddle point.

(b) For $f(x,y) = (x+y)^3 - (x-y)(x-5y)$, setting $\nabla f(x,y) = \mathbf{0}$ gives the system:

$$\begin{aligned} 0 &= f_1(x,y) = 3(x+y)^2 - (x-5y) - (x-y) = 3(x+y)^2 - 2x + 6y \\ 0 &= f_2(x,y) = 3(x+y)^2 + (x-5y) - (x-y)(-5) = 3(x+y)^2 + 6x - 10y \end{aligned}$$

We *could* solve one equation for y in terms of x and substitute it into the other, but if we subtract the equations first, we eliminate the $3(x+y)^2$ term:

$$0 = -8x + 16y$$

Solving this for x gives $x = 16y/8 = 2y$, and substituting this into one of the original equations, say the first, will eliminate x:

$$0 = 3(2y + y)^2 - 2(2y) + 6y = 27y^2 + 2y = y(27y + 2)$$

This has two solutions: $y = 0$ and $y = -2/27$. Since we know $x = 2y$, this gives the two solutions $(0,0)$ and $(-4/27, -2/27)$.

To classify these points using the second derivative test, we calculate the Hessian at an arbitrary point (x, y) as:

$$\mathcal{H}(x, y) = \begin{bmatrix} 6(x+y) - 2 & 6(x+y) + 6 \\ 6(x+y) + 6 & 6(x+y) - 10 \end{bmatrix}$$

For critical point $(0, 0)$, we see

$$\mathcal{H}(0, 0) = \begin{bmatrix} -2 & 6 \\ 6 & -10 \end{bmatrix}$$

which has $\det(\mathcal{H}) = -16$, so $(0, 0)$ is a saddle point.

For critical point $(-4/27, -2/27)$, we see

$$\mathcal{H}(-4/27, -2/27) = \begin{bmatrix} -4/3 - 2 & -4/3 + 6 \\ -4/3 + 6 & -4/3 - 10 \end{bmatrix} = \begin{bmatrix} -10/3 & 14/3 \\ 14/3 & -34/3 \end{bmatrix}$$

which has $\det(\mathcal{H}) = 16$ with top-left element $-10/3 < 0$, so $(-4/27, -2/27)$ is a local maximum.

(c) For $f(x, y) = e^{-x^2-y^2}(1 - e^{x^2})$, setting $\nabla f(x, y) = \mathbf{0}$ gives the system

$$\begin{aligned} 0 = f_1(x, y) &= -2xe^{-x^2-y^2}(1 - e^{x^2}) + e^{-x^2-y^2}(-2xe^{x^2}) = -2xe^{-x^2-y^2} \\ 0 = f_2(x, y) &= -2ye^{-x^2-y^2}(1 - e^{x^2}) \end{aligned}$$

Since the exponential function is never 0, the only solution to the first equation must be $x = 0$. Substituting this into the second equation gives

$$0 = -2ye^{-y^2}(1 - 1) = 0$$

but this equation is satisfied for *all* y, so all points of the form $(0, b)$ for any b are critical.

If we try to use the second derivative test to classify these points, we discover that $\det(\mathcal{H}) = 0$, so the test is inconclusive. We must resort to the brute-force method, looking at changes in f for small changes dx and dy away from the critical point. For the critical point $(0, b)$, we have

$$\begin{aligned} \Delta f &= f(dx, b + dy) - f(0, b) \\ &= e^{-dx^2-(b+dy)^2}\left(1 - e^{-dx^2}\right) - e^{-b^2}\left(1 - e^0\right) \\ &= e^{-(b+dy)^2}\left(e^{-dx^2} - 1\right) \end{aligned}$$

But $e^{-(b+dy)^2}$ is always positive while $dx^2 \geq 0$ ensures that $e^{-dx^2} \leq 1$. Therefore, $\Delta f \leq 0$ for all small dx and dy from which it follows that all points $(0, b)$ are local maxima.

(d) For $f(x, y) = 2 \sin x \cos y$, setting $\nabla f(x, y) = \mathbf{0}$ gives the system:

$$\begin{aligned} 0 = f_1(x, y) &= 2 \cos x \cos y \\ 0 = f_2(x, y) &= -2 \sin x \sin y \end{aligned}$$

The first equation will be satisfied whenever $\cos x = 0$ or $\cos y = 0$. Let us consider these cases separately:

- Case 1: $\cos x = 0$

  When $\cos x = 0$, that implies $\sin x = \pm 1$. Therefore, the second equation $0 = -2 \sin x \sin y$ forces $\sin y = 0$. Thus, the critical points for this case are all those (x, y) where $\cos x = 0$ and $\sin y = 0$ or, in other words, the points $(\pi/2 + m\pi, n\pi)$ for m, n any integers.

- Case 2: $\cos y = 0$

  Similarly, this implies $\sin y = \pm 1$, so the second equation $0 = -2 \sin x \sin y$ forces $\sin x = 0$, and the critical points for this case are those where $\cos y = 0$ and $\sin x = 0$ or, in other words, the points $(m\pi, \pi/2 + n\pi)$ for m, n any integers.

To classify these points, we calculate the Hessian at an arbitrary (x, y):

$$\mathcal{H}(x,y) = \begin{bmatrix} -2\sin x\cos y & -2\cos x\sin y \\ -2\cos x\sin y & -2\sin x\cos y \end{bmatrix}$$

Now, we have to carefully examine all the possible cases and subcases.

For points of the form $(\pi/2 + m\pi, n\pi)$, we have, as mentioned above, $\cos x = 0$ and $\sin y = 0$, but the Hessian also depends on $\sin x = \pm 1$ and $\cos y = \pm 1$, so we have to consider all four subcases:

- Subcase 1(a): $\sin x = 1$, $\cos x = 0$, $\sin y = 0$, $\cos y = 1$

  These are the points $(\pi/2 + 2\pi m, 2\pi n)$ for all integers m and n, and the Hessian evaluates to:
  $$\mathcal{H} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$
  which has $\det(\mathcal{H}) = 4$ with top-left element $-2 < 0$, so these are local maxima.

- Subcase 1(b): $\sin x = 1$, $\cos x = 0$, $\sin y = 0$, $\cos y = -1$

  These are the points $(\pi/2+2\pi m, \pi+2\pi n)$ for all integers m and n, and the Hessian evaluates to:
  $$\mathcal{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
  which has $\det(\mathcal{H}) = 4$ with top-left element $2 > 0$, so these are local minima.

- Subcase 1(c): $\sin x = -1$, $\cos x = 0$, $\sin y = 0$, $\cos y = 1$

  These are the points $(3\pi/2 + 2\pi m, 2\pi n)$ for all integers m and n, and the Hessian evaluates to:
  $$\mathcal{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
  which has $\det(\mathcal{H}) = 4$ with top-left element $2 > 0$, so these are local minima

- Subcase 1(d): $\sin x = -1$, $\cos x = 0$, $\sin y = 0$, $\cos y = -1$

  These are the points $(3\pi/2+2\pi m, \pi+2\pi n)$ for all integers m and n, and the Hessian evaluates to:
  $$\mathcal{H} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$
  which has $\det(\mathcal{H}) = 4$ with top-left element $-2 < 0$, so these are local maxima.

For points of the form $(m\pi, \pi/2 + n\pi)$, we have, as mentioned above, $\cos y = 0$ and $\sin x = 0$. Even though the Hessian also depends on $\sin y = \pm 1$ and $\cos x = \pm 1$, it's not hard to see that we'll always have:

$$\mathcal{H} = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \qquad \text{or} \qquad \mathcal{H} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

In either case, $\det(\mathcal{H}) = -4$, so all points of this form are saddle points.

2. Find and classify all critical and singular points of $f(x, y) = 7\sqrt{x^2 + y^2} - 2(x - 1)^2 + (x + 1)^2$.

The gradient is given by

$$\nabla f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} \frac{7x}{\sqrt{x^2+y^2}} - 2x + 6 \\ \frac{7y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

Thus, $\nabla f(x, y)$ does not exist iff $x^2 + y^2 \leq 0$ iff $(x, y) = (0, 0)$, so the only singular point is $(0, 0)$.

Setting $\nabla f(x, y) = \mathbf{0}$ gives the system:

$$0 = \frac{7x}{\sqrt{x^2 + y^2}} - 2x + 6$$
$$0 = \frac{7y}{\sqrt{x^2 + y^2}}$$

The second equation implies $y = 0$, and substituting into the first equation gives

$$0 = \frac{7x}{\sqrt{x^2}} - 2x + 6 = \frac{7x}{|x|} - 2x + 6$$

Considering the two cases separately:

- Case 1: $x < 0$. Then $0 = -7 - 2x + 6 = -1 - 2x$ giving $x = -1/2$.
- Case 2: $x \geq 0$. Then $0 = +7 - 2x + 6 = 13 - 2x$ giving $x = 13/2$.

gives the two critical points $(-1/2, 0)$ and $(13/2, 0)$.

To classify the critical points, we calculate the second-order partials:

$$f_{11} = \frac{7\sqrt{x^2 + y^2} - \frac{7x^2}{\sqrt{x^2+y^2}}}{x^2 + y^2} - 2 = \frac{7y^2}{(x^2 + y^2)^{3/2}} - 2$$

$$f_{12} = \frac{-7xy}{(x^2 + y^2)^{3/2}}$$

$$f_{22} = \frac{7\sqrt{x^2 + y^2} - \frac{7y^2}{\sqrt{x^2+y^2}}}{x^2 + y^2} = \frac{7x^2}{(x^2 + y^2)^{3/2}}$$

which give the Hessian matrices

$$\mathcal{H}(-1/2, 0) = \begin{bmatrix} -2 & 0 \\ 0 & 14 \end{bmatrix} \qquad \mathcal{H}(13/2, 0) = \begin{bmatrix} -2 & 0 \\ 0 & 14/13 \end{bmatrix}$$

both of which have negative determinant. Therefore, both $(-1/2, 0)$ and $(13/2, 0)$ are saddle points.

The singular point $(0, 0)$ must be classified by considering small movements dx and dy away from the point:

$$\Delta f = f(dx, dy) - f(0, 0) = 7\sqrt{dx^2 + dy^2} - 2(dx - 1)^2 + (dx + 1)^2 + 1$$
$$= 7\sqrt{dx^2 + dy^2} + (6 - dx)dx$$

But, $7\sqrt{dx^2 + dy^2} \geq 7\sqrt{dx^2} = 7|dx| \geq -(6 - dx)dx$ for all small dx whether positive or negative. Therefore, $\Delta f \geq 0$ for all small dx and dy, and $(0, 0)$ is a local minimum.

3. Find the (minimum) distance between the parabolas $\mathbf{r}_1(t) = \langle 0, 2t, -t^2 \rangle$, $-\infty < t < \infty$ and $\mathbf{r}_2(u) = \langle -u, 3, u^2 \rangle$, $-\infty < u < \infty$.

Let $f(t, u)$ be the squared distance between the points $\mathbf{r}_1(t)$ and $\mathbf{r}_2(u)$. It is given by

$$f(t, u) = (0 + u)^2 + (2t - 3)^2 + (-t^2 - u^2)^2 = u^2 + (2t - 3)^2 + (t^2 + u^2)^2$$

To find the t and u that minimize f, set $\nabla f(t, u) = \mathbf{0}$ to find the critical points:

$$\begin{aligned} 0 &= 4(2t - 3) + 4t(t^2 + u^2) = 8t - 12 + 4t^3 + 4tu^2 \\ 0 &= 2u + 4u(t^2 + u^2) = 2u(1 + 2t^2 + 2u^2) \end{aligned}$$

Since $(1 + 2t^2 + 2u^2) \geq 1 > 0$, the second equation implies $u = 0$. Substituting into the first, we have

$$0 = 8t - 12 + 4t^3$$

By inspection, this has a solution $t = 1$, so the right-hand side must be divisible by $(t - 1)$. Using long division, we get the factorization:

$$0 = 8t - 12 + 4t^3 = 4(t - 1)(t^2 + t + 3)$$

and since t^2+t+3 has no real roots, the only solution is $t = 1$, giving the critical point $(t, u) = (1, 0)$.

We could use a Hessian test (the Hessian is $\mathcal{H}(1, 0) = \left[\begin{smallmatrix} 20 & 0 \\ 0 & 6 \end{smallmatrix}\right]$) to verify that this critical point is a local minimum, but common sense tells us that there must *be* a minimum distance between the curves, and it must occur at either a singular point (there are none), a boundary point (there are none), or a critical point (there is one), so the critical point must give the minimum distance.

At the critical point $(1, 0)$, the squared distance is $f(1, 0) = 2$, so the minimum distance between the curves is $\sqrt{2}$.

4. For what values of the constant k does the function $f(x, y) = kx^3 + x^2 + 2y^2 - 4x - 4y$ have: (a) no critical points; (b) exactly one critical point; (c) exactly two critical points? For parts (b) and (c), give the critical points (in terms of k).

For a fixed k, setting $\nabla f(x, y) = \mathbf{0}$ gives the system:

$$
\begin{aligned}
0 &= 3kx^2 + 2x - 4 \\
0 &= 4y - 4
\end{aligned}
$$

The second equation doesn't depend on k and always has solution $y = 1$. For the first equation, we should first deal with the special case where $k = 0$ and the equation is linear: $0 = 2x - 4$ which has solution $x = 2$, giving the single critical point $(x, y) = (2, 1)$.

For $k \neq 0$, the second equation is quadratic with discriminant $b^2 - 4ac = 2^2 - 4(3k)(-4) = 4 + 48k$. If $k < -1/12$, then the discriminant is negative, the equation has no real solutions, and f has no critical points. If $k = -1/12$, then the discriminant is zero, and the equation has the single solution:

$$x = \frac{-2}{2(3k)} = 4$$

giving the single critical point $(4, 1)$. If $k > -1/12$, then the discriminant is positive, and the equation has two solutions

$$x = \frac{-2 \pm \sqrt{4 + 48k}}{6k}$$

giving the two critical points

$$(x, y) = \left(\frac{-1 + \sqrt{1 + 12k}}{3k}, 1 \right) \qquad (x, y) = \left(\frac{-1 - \sqrt{1 + 12k}}{3k}, 1 \right)$$

To sum up, the answers are

(a) For $k < -1/12$, there are no critical points;

(b) For $k = 0$, there is the single critical point $(2, 1)$, and for $k = -1/12$, there is the single critical point $(4, 1)$;

(c) For $k > -1/12$ but $\neq 0$, there are two critical points

$$(x, y) = \left(\frac{-1 + \sqrt{1 + 12k}}{3k}, 1 \right) \qquad (x, y) = \left(\frac{-1 - \sqrt{1 + 12k}}{3k}, 1 \right)$$

5. Suppose the outside air temperature is given by

$$T(x,y,z) = -40 + \left(60 + 90z + \frac{5}{20 + x^2 + xy + y^2 - 2x + 4y}\right) e^{-z}$$

for $z \geq 0$ (where $z = 0$ represents ground level). (a) Find any critical points. (b) Can a point at ground level have a global minimum or maximum temperature value? Why or why not? (c) Find the points of global minimum and maximum temperature value, or explain why such points do not exist.

(a) Setting $\nabla T(x,y,z) = \mathbf{0}$ gives the system:

$$\begin{aligned} 0 &= \frac{-5(2x + y - 2)e^{-z}}{(20 + x^2 + xy + y^2 - 2x + 4y)^2} \\ 0 &= \frac{-5(x + 2y + 4)e^{-z}}{(20 + x^2 + xy + y^2 - 2x + 4y)^2} \\ 0 &= \left(30 - 90z - \frac{5}{20 + x^2 + xy + y^2 - 2x + 4y}\right) e^{-z} \end{aligned}$$

The first two equations imply $2x + y - 2 = 0$ and $x + 2y + 4 = 0$. Solving the former for y gives $y = 2 - 2x$, and substituting into the latter gives $x = 8/3$ and so $y = 2 - 2(8/3) = -10/3$. Thus, the third equation gives

$$z = \frac{1}{90}\left(30 - \frac{5}{20 + x^2 + xy + y^2 - 2x + 4y}\right) = \frac{21}{64}$$

Therefore, the single critical point is $(8/3, -10/3, 21/64)$.

(b) Let us begin by considering the denominator

$$g(x,y) = 20 + x^2 + xy + y^2 - 2x + 4y$$

Setting $\nabla g(x,y) = \mathbf{0}$ and solving gives a single critical point $(x,y) = (8/3, -10/3)$. Since

$$\begin{aligned} \Delta g &= g(8/3 + dx, -10/3 + dy) - g(8/3, -10/3) \\ &= dx^2 + dx\,dy + dy^2 = (dx + dy/2)^2 + 3dy^2/4 \geq 0 \end{aligned}$$

(or by the second derivative test) we conclude that $(8/3, -10/3)$ is a point of local minimum value. Since there are no singular or boundary points, $g(8/3, -10/3) = 32/3$ is the global minimum value of the denominator g.

Now, at ground level, the temperature is given by:

$$T(x,y,0) = 20 + \frac{5}{g(x,y)}$$

As $g(x,y)$ can be made arbitrarily large (just pick x and y of large magnitude), there is no minimum ground-level temperature. The maximum ground-level temperature is attained where $g(x,y)$ is at a minimum, at the critical point $(8/3, -10/3)$. This is the only possible ground-level point where we might have a global extreme value, so let us fix $(x,y) = (8/3, -10/3)$ and let $z \geq 0$ vary, observing the temperature T along a vertical line from ground level upwards as a function of z alone:

$$T(z) = -40 + (60 + 90z + 15/32)e^{-z} \tag{1}$$

Note that $T(0) = -40 + (60 + 15/32) = 655/32 \approx 20.47$ at ground level. Setting $0 = T'(z) = (30 - 90z - 15/32)e^{-z}$ gives the critical point $z_0 = 21/64$ (with $T''(z_0) = (-120 + 90z_0 + 15/32)e^{-z_0} \approx -64.8$ indicating it is a point of local maximum temperature along the vertical line). In particular, $T(21/64) \approx 24.82$, so the maximum ground-level temperature $T(0) \approx 20.47$ is clearly not the global maximum temperature.

(c) Since $g(x, y)$ is positive, we have $T(x, y, z) = -40 + (60 + 90z + 5/g(x, y))e^{-z} > -40$ everywhere. However, by taking z large, we can bring $T(x, y, z)$ arbitrarily close to -40. Therefore, $T(x, y, z)$ has no global minimum.

Also, we have established that none of the ground-level (boundary) points are global maxima. Since there are no singular points (as the denominator g is never zero), if we can show that the critical point $(8/3, -10/3, 21/64)$ is a local maximum, then it will be the global maximum. This can be done via a Hessian test (the eigenvalues are -64.8, -0.0950, and -0.0317), but we could also just observe that $T(x, y, z) = -40 + (60 + 90z + 5/g(x, y))e^{-z}$ is always maximized, for any fixed z, by the values $(x, y) = (8/3, -10/3)$ that minimize $g(x, y)$, so we need only consider the temperatures given by formula (1) along the vertical line $(x, y) = (8/3, -10/3)$ from the ground upwards. And, we already showed that the temperature attained its maximum value along this line at $z = 21/64$.

Therefore, $(8/3, -10/3, 21/64)$ is the point of global maximum temperature $T \approx 24.82$ and there is no point of global minimum temperature.

6. [MATLAB] Consider the function $f(x, y, z) = \sin(xe^z)\cos y$.

1. Find algebraic expressions for the gradient $\nabla f(x, y, z)$ and the Hessian $\mathcal{H}(x, y, z)$.
2. Define the point $(a, b, c) = (\pi/e, \pi/2, 1)$, and show it is a critical point.
3. Using MATLAB, find the eigenvalues (and corresponding eigenvectors) of $\mathcal{H}(a, b, c)$, and verify (a, b, c) is a saddle point.
4. Find a vector (u, v, w) of length 0.1 parallel to one of the eigenvectors of $\mathcal{H}(a, b, c)$ such that $f(a + u, b + v, c + w) < f(a, b, c)$. [Check the inequality using MATLAB.]
5. Find a vector (u', v', w') of length 0.1 parallel to one of the eigenvectors of $\mathcal{H}(a, b, c)$ such that $f(a + u', b + v', c + w') > f(a, b, c)$. [Check the inequality using MATLAB.]

(a) For $f(x, y, z) = \sin(xe^z)\cos(y)$, we have

$$
\begin{aligned}
f_1 &= e^z\cos(xe^z)\cos(y); & f_{11} &= -e^{2z}\sin(xe^z)\cos(y),\\
& & f_{12} &= -e^z\cos(xe^z)\sin(y),\\
& & f_{13} &= e^z\cos(xe^z)\cos(y) - xe^{2z}\sin(xe^z)\cos(y),\\
f_2 &= -\sin(xe^z)\sin(y); & f_{21} &= -e^z\cos(xe^z)\sin(y),\\
& & f_{22} &= -\sin(xe^z)\cos(y),\\
& & f_{23} &= -xe^z\cos(xe^z)\sin(y),\\
f_3 &= xe^z\cos(xe^z)\cos(y); & f_{31} &= e^z\cos(xe^z)\cos y - xe^{2z}\sin(xe^z)\cos(y),\\
& & f_{32} &= -xe^z\cos(xe^z)\sin(y),\\
& & f_{33} &= xe^z\cos(xe^z)\cos(y) - x^2e^{2z}\sin(xe^z)\cos(y).
\end{aligned}
$$

These are the components that go into

$$
\operatorname{grad} f(x, y, z) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \qquad \mathcal{H}(x, y, z) = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}.
$$

Notice that matrix $\mathcal{H}$ is symmetric.

(b) Let $\mathbf{a} = (a, b, c) = (\pi/e,\ \pi/2,\ 1)$. At this point, $xe^z = ae^c = \pi$, so

$$
\operatorname{grad} f(x, y, z) = \begin{bmatrix} e\cos(\pi)\cos(\pi/2) \\ -\sin(\pi)\sin(\pi/2) \\ \pi\cos(\pi)\cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

so $\mathbf{a}$ is indeed a critical point. Likewise,

$$
\mathcal{H}(\mathbf{a}) = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} 0 & e & 0 \\ e & 0 & \pi \\ 0 & \pi & 0 \end{bmatrix}.
$$

(c) The MATLAB command

```
H = [0,exp(1),0; exp(1),0,pi; 0, pi, 0]
```

defines the Hessian of interest, and then the command `[V,D] = eig(H)` gives

$$
V = \begin{bmatrix} 0.4627 & -0.7562 & -0.4627 \\ 0.7071 & 0 & 0.7071 \\ 0.5347 & 0.6543 & -0.5347 \end{bmatrix}, \qquad D = \begin{bmatrix} 4.1544 & 0 & 0 \\ 0 & 0.0000 & 0 \\ 0 & 0 & -4.1544 \end{bmatrix}.
$$

Here the eigenvalues of H appear as diagonal entries of D: since both positive and negative eigenvalues appear, $\mathbf{a}$ is indeed a saddle point.

(d) The negative eigenvalue in column 3 of matrix D above indicates that the unit vector in column 3 of matrix V above is a second-order descent direction for f at $\mathbf{a}$. A vector of length 0.1 in this direction is

$$(u, v, w) = (-0.04627,\ 0.07071,\ -0.05347).$$

The exact value of $f(a, b, c)$ is 0, but rounding errors in MATLAB lead to

```
≫ a=pi/exp(1); b=pi/2; c=1;
≫ x=a;y=b;z=c;f=sin(x*exp(z))*cos(y)
f =
  7.4983e-033
≫ u=-0.04627; v=0.07071; w=-0.05347;
≫ x=a+u;y=b+v;z=c+w;f=sin(x*exp(z))*cos(y)
f =
   -0.0197
```

Thus, indeed, $f(a + u, b + v, c + w) < f(a, b, c) = 0$.

(e) The positive eigenvalue in column 1 of matrix D above indicates that the unit vector in column 1 of matrix V above is a second-order ascent direction for f at $\mathbf{a}$. A vector of length 0.1 in this direction is $(u', v', w') = (0.04627,\ 0.07071,\ 0.05347)$. Switching to MATLAB (where primes are not allowed in variable names),

```
≫ a=pi/exp(1); b=pi/2; c=1;
≫ u=0.04627; v=0.07071; w=0.05347;
≫ x=a+u; y=b+v; z=c+w; f=sin(x*exp(z))*cos(y)
f =
   0.0212
```

Thus, indeed, $f(a + u', b + v', c + w') > f(a, b, c) = 0$.

7. [MATLAB] For each of the following functions (i) find algebraic expressions for the gradient ∇f and the Hessian $\mathcal{H}$; (ii) write a MATLAB script that implements Newton's method for finding critical points; (iii) run your script with each of the given starting points and include in your answer the results of the first 10 iterations; and (iv) in those cases where the method appears to be converging, give a probable classification of the critical point.

(a) $e^{-(x-1)^2-y^2} - e^{-(x+1)^2-y^2}$ starting at $(-1.5, 0.1)$, $(1.5, 0.1)$, and $(1.5, 0.2)$

(b) $(y^2+z^2-3)^2+(x^2+z^2-2)^2+(x^2-z)^2$ starting at $(1, 1.5, 1)$, $(0.5, 0.5, 0.5)$, and $(0.1, -0.1, 0.1)$

(a) For $f(x,y) = e^{-(x-1)^2-y^2} - e^{-(x+1)^2-y^2} = \left[e^{-(x-1)^2} - e^{-(x+1)^2}\right] e^{-y^2}$, we have

$$f_1 = \left[-2(x-1)e^{-(x-1)^2} + 2(x+1)e^{-(x+1)^2}\right] e^{-y^2},$$

$$f_{11} = \left[\left(-2+4(x-1)^2\right) e^{-(x-1)^2} + \left(2-4(x+1)^2\right) e^{-(x+1)^2}\right] e^{-y^2},$$

$$f_{12} = \left[-2(x-1)e^{-(x-1)^2} + 2(x+1)e^{-(x+1)^2}\right] \left(-2ye^{-y^2}\right),$$

$$f_2 = \left[e^{-(x-1)^2} - e^{-(x+1)^2}\right] \left(-2ye^{-y^2}\right),$$

$$f_{21} = \left[-2(x-1)e^{-(x-1)^2} + 2(x+1)e^{-(x+1)^2}\right] \left(-2ye^{-y^2}\right),$$

$$f_{22} = \left[e^{-(x-1)^2} - e^{-(x+1)^2}\right] \left(-2[1-2y^2]e^{-y^2}\right).$$

Ten steps of Newton's Method from three starting points gave the results below.

```
>> newton7a([-1.5; 0.1])

 k     x(k)         y(k)
 0  -1.500e+000  1.000e-001
 1  -5.282e-001 -1.002e-001
 2  -1.213e+000  1.066e-001
 3  -1.008e+000 -1.130e-002
 4  -1.033e+000  1.862e-005
 5  -1.033e+000 -5.902e-013
 6  -1.033e+000  0.000e+000
 7  -1.033e+000  0.000e+000
 8  -1.033e+000  0.000e+000
 9  -1.033e+000  0.000e+000
10  -1.033e+000  0.000e+000

Report on point #10:
Objective  gradient:      1.4e-017  0.0e+000
Hessian eigenvalues:      2.2e+000  2.0e+000
```

At $(-1.033, 0.000)$, grad $f \approx \mathbf{0}$ and both Hessian eigenvalues are positive, so this point is probably a **local minimizer**.

```
>> newton7a([1.5; 0.1])

 k     x(k)         y(k)
 0   1.500e+000  1.000e-001
 1   5.282e-001 -1.002e-001
 2   1.213e+000  1.066e-001
 3   1.008e+000 -1.130e-002
 4   1.033e+000  1.862e-005
```

```
  5   1.033e+000 -5.902e-013
  6   1.033e+000  0.000e+000
  7   1.033e+000  0.000e+000
  8   1.033e+000  0.000e+000
  9   1.033e+000  0.000e+000
 10   1.033e+000  0.000e+000

Report on point #10:
Objective  gradient:      1.4e-017  0.0e+000
Hessian eigenvalues:    -2.2e+000 -2.0e+000
```

At $(1.033, 0.000)$, grad $f \approx \mathbf{0}$ and both Hessian eigenvalues are negative, so this point is probably a **local maximizer**.

```
≫ newton7a([1.5; 0.2])

  k     x(k)        y(k)
  0   1.500e+000  2.000e-001
  1   3.956e-001 -2.551e-001
  2   2.208e+000  1.231e+000
  3   2.452e+000  1.480e+000
  4   2.644e+000  1.675e+000
  5   2.808e+000  1.842e+000
  6   2.954e+000  1.991e+000
  7   3.089e+000  2.128e+000
  8   3.213e+000  2.255e+000
  9   3.330e+000  2.374e+000
 10   3.440e+000  2.486e+000

Report on point #10:
Objective  gradient:    -2.6e-005 -2.7e-005
Hessian eigenvalues:    -1.1e-005  2.5e-004
```

It does not look like Newton's Method is converging in this case.

(b) For $f(x, y, z) = (y^2 + z^2 - 3)^2 + (x^2 + z^2 - 2)^2 + (x^2 - z)^2$, we have

$$f_1 = 4x(x^2 + z^2 - 2) + 4x(x^2 - z) = 8x^3 + 4xz^2 - 4xz - 8x,$$
$$f_{11} = 24x^2 + 4z^2 - 4z - 8,$$
$$f_{12} = 0,$$
$$f_{13} = 8xz - 4x,$$
$$f_2 = 4y(y^2 + z^2 - 3) = 4y^3 + 4yz^2 - 12y,$$
$$f_{21} = 0,$$
$$f_{22} = 12y^2 + 4z^2 - 12,$$
$$f_{23} = 8yz,$$
$$f_3 = 4z(y^2 + z^2 - 3) + 4z(x^2 + z^2 - 2) - 2(x^2 - z) = 4x^2z + 4y^2z + 8z^3 - 18z - 2x^2,$$
$$f_{31} = 8xz - 4x,$$
$$f_{32} = 8yz,$$
$$f_{33} = 4x^2 + 4y^2 + 24z^2 - 18.$$

Ten steps of Newton's Method from three starting points gave the results below.

```
≫ newton7b([1; 1.5; 1])

  k     x(k)        y(k)        z(k)
  0   1.000e+000  1.500e+000  1.000e+000
  1   1.001e+000  1.424e+000  9.949e-001
  2   1.000e+000  1.414e+000  9.999e-001
```

```
  3   1.000e+000  1.414e+000  1.000e+000
  4   1.000e+000  1.414e+000  1.000e+000
  5   1.000e+000  1.414e+000  1.000e+000
  6   1.000e+000  1.414e+000  1.000e+000
  7   1.000e+000  1.414e+000  1.000e+000
  8   1.000e+000  1.414e+000  1.000e+000
  9   1.000e+000  1.414e+000  1.000e+000
 10   1.000e+000  1.414e+000  1.000e+000

Report on point #10:
Objective  gradient:      0.0e+000  0.0e+000  0.0e+000
Hessian eigenvalues:      1.6e+001  5.0e+000  2.9e+001
```

At $(1.000, 1.414, 1.000)$, $\operatorname{grad} f \approx \mathbf{0}$ and the Hessian eigenvalues are all positive, so this point is probably a **local minimizer**.

```
≫ newton7b([0.5; 0.5; 0.5])

  k     x(k)        y(k)        z(k)
  0   5.000e-001  5.000e-001  5.000e-001
  1  -6.667e-001 -3.553e-001 -4.211e-001
  2  -1.195e+000  6.921e-002 -1.554e-001
  3  -1.015e+000 -3.209e-004 -1.420e-001
  4  -9.661e-001  3.255e-007 -1.314e-001
  5  -9.623e-001 -1.644e-011 -1.308e-001
  6  -9.623e-001  5.463e-018 -1.308e-001
  7  -9.623e-001 -5.873e-029 -1.308e-001
  8  -9.623e-001  0.000e+000 -1.308e-001
  9  -9.623e-001  0.000e+000 -1.308e-001
 10  -9.623e-001  0.000e+000 -1.308e-001

Report on point #10:
Objective  gradient:     0.0e+000  0.0e+000  2.2e-016
Hessian eigenvalues:   -1.2e+001  1.6e+001 -1.5e+001
```

At $(-0.962, 0.000, -0.131)$, $\operatorname{grad} f \approx \mathbf{0}$ and the Hessian has both positive and negative eigenvalues, so this is probably a **saddle point**.

```
≫ newton7b([0.1; -0.1; 0.1])

  k     x(k)        y(k)        z(k)
  0   1.000e-001 -1.000e-001  1.000e-001
  1   1.999e-003  1.356e-003 -7.210e-004
  2  -7.384e-007 -2.133e-009  4.465e-007
  3  -1.648e-013  2.837e-022  6.058e-014
  4  -4.998e-027  0.000e+000  3.004e-027
  5  -7.175e-043  0.000e+000 -3.587e-043
  6  -7.965e-059  0.000e+000  7.965e-059
  7  -8.843e-075  0.000e+000 -1.769e-074
  8  -9.818e-091  0.000e+000  3.927e-090
  9  -1.090e-106  0.000e+000 -8.720e-106
 10  -1.210e-122  0.000e+000  1.936e-121

Report on point #10:
Objective  gradient:     9.7e-122  0.0e+000 -3.5e-120
Hessian eigenvalues:   -1.2e+001 -8.0e+000 -1.8e+001
```

At $(0, 0, 0)$, $\operatorname{grad} f \approx \mathbf{0}$ and the Hessian eigenvalues are all negative, so this point is probably a **local maximizer**.

8. Recall that a function $f(x, y)$ is *harmonic* if it satisfies $f_{11}(x, y) + f_{22}(x, y) = 0$ for all x and y in its domain. Suppose f is a harmonic function with domain all of $\mathbb{R}^2$ and with $f_{11}(x, y) \neq 0$ for all x and y. Prove that f has no local minima or maxima.

[All Theorems mentioned below are from Section 13.1.]

Let f be a function satisfying the hypotheses stated in the question. By Theorem 1, any local minimum or maximum must be a critical point of f, a singular point of f, or a boundary point of the domain of f. Now, because f is harmonic, its second-order derivatives exist at all points in its domain, so its first-order derivatives (and so its gradient) exist for all points in the domain. Therefore, there are no singular points. Also, because f's domain is all of $\mathbb{R}^2$, there are no boundary points. Therefore, any local minimum or maximum must be a critical point. We will now show that all critical points are saddle points which will allow us to conclude that f has no local minima or maxima.

Let (a, b) be a critical point of f. Applying Theorem 3, the second derivative test, note that the Hessian at the critical point (a, b) is:

$$\mathcal{H} = \begin{bmatrix} f_{11}(a, b) & f_{12}(a, b) \\ f_{12}(a, b) & f_{22}(a, b) \end{bmatrix}$$

so we have $\det(\mathcal{H}) = f_{11}(a, b)f_{22}(a, b) - f_{12}(a, b)^2$. However, $f_{11}(a, b) \neq 0$ by assumption, and because f is harmonic, we have $f_{22}(a, b) = -f_{11}(a, b)$. Therefore, $\det(\mathcal{H}) = -f_{11}(a, b)^2 - f_{12}(a, b)^2 \leq -f_{11}(a, b)^2 < 0$. Therefore, $\mathcal{H}$ is indefinite, and (a, b) is a saddle point.