Math 263 Assignment #3 Solutions

1. A function $z = f(x, y)$ is called *harmonic* if it satisfies Laplace's equation:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Determine whether or not the following are harmonic.

(a) $z = \sqrt{x^2 + y^2}$.

We use the one-variable chain rule to compute the partials of z with respect x and y.

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \text{ and } \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

We only needed to compute one of these partials and then use the symmetry of $z = \sqrt{x^2 + y^2}$ to compute the other. Applying the one-variable quotient rule we see that

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) = \frac{\sqrt{x^2 + y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{x^2 + y^2} = \frac{(x^2 + y^2) - x^2}{(x^2 + y^2)^{3/2}} = \frac{y^2}{(x^2 + y^2)^{3/2}}.$$

By symmetry, we see that

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2}{(x^2 + y^2)^{3/2}}.$$

Therefore $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 1/\sqrt{x^2 + y^2} \neq 0$, so $z = \sqrt{x^2 + y^2}$ is not harmonic.

(b) $z = e^{-x} \sin y$.

These partials are easier to compute than in (a): $\partial z/\partial x = -e^{-x} \sin y$, $\partial^2 z/\partial x^2 = e^{-x} \sin y$, $\partial z/\partial y = e^{-x} \cos y$, and $\partial^2 z/\partial y^2 = -e^{-x} \sin y$. Therefore the function is harmonic.

(c) $z = 3x^2 y - y^3$.

Again we compute the partials: $\partial z/\partial x = 6xy$, $\partial^2 z/\partial x^2 = 6y$, $\partial z/\partial y = 3x^2 - 3y^2$, and $\partial^2 z/\partial y^2 = -6y$. Since $6y - 6y = 0$, this function is also harmonic.

2. Give an example of a function with the indicated properties or show that no such function exists.

(a) A function $f(x, y)$ with continuous first and second order partial derivatives in the xy-plane and which satisfies $\partial f/\partial x = 6xy^2$ and $\partial f/\partial y = 8x^2 y$.

The simplest way to solve this problem is with Theorem 1 from section 12.4, because then the mixed second-order partial derivatives must be equal, i.e., we should have $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$. For the given f, we have that

$$\frac{\partial^2 f}{\partial x\partial y} = 16xy \neq 12xy = \frac{\partial^2 f}{\partial y\partial x},$$

so there can be no such function f.

(b) A function $g(x, y)$ satisfying the equations $\partial g/\partial x = \partial g/\partial y = 2xy$.

For part (b), we could make the same argument as in (a), as long as we verify that g has continuous first and second-order partials everywhere. Since $2x \neq 2y$, we could conclude that no g exists. I don't feel like verifying the hypotheses of Theorem 1 of section 12.4. Let's try a different method. Note that

$$g(x,y) = \int \frac{\partial g}{\partial x} dx = \int 2xy dx = x^2 y + h(y).$$

Now taking the partial of $g(x,y)$ with respect to y gives that $\partial g/\partial y = x^2 + h'(y)$. But by our hypothesis, we have that $\partial g/\partial y = 2xy$. However, $x^2 + h'(y)$ cannot equal $2xy$, since h' is a function of y only. Thus there are no functions g with the indicated properties.

3. Use the appropriate version of the chain rule to compute the following.

(a) dw/dt at $t = 3$, where $w = \ln(x^2 + y^2 + z^2)$, $x = \cos t$, $y = \sin t$, and $z = 4\sqrt{t}$.

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= \frac{-2x \sin t}{x^2 + y^2 + z^2} + \frac{2y \cos t}{x^2 + y^2 + z^2} + \frac{4zt^{-1/2}}{x^2 + y^2 + z^2} \\ &= \frac{-2 \cos t \sin t + 2 \sin t \cos t + 16}{\cos^2 t + \sin^2 t + 16t} \\ &= \frac{16}{1 + 16t} \end{aligned}$$

Therefore $\left.\dfrac{dw}{dt}\right|_{t=3} = \dfrac{16}{49}$.

(b) $\partial z/\partial u$ and $\partial z/\partial v$, where $z = xy$, $x = u \cos v$, and $y = u \sin v$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = y \cos v + x \sin v = 2u \sin v \cos v$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -yu \sin v + xu \cos v = u^2(\cos^2 v - \sin^2 v)$$

4. Suppose that a duck is swimming in the circle $x = \cos t$, $y = \sin t$ and that the water temperature is given by the formula $T = x^2 e^y - xy^3$. Find the rate of change in temperature the duck might feel.

We want to compute dT/dt. This is just a simple application of the chain-rule.

$$\begin{aligned} \frac{dT}{dt} &= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = (2xe^y - y^3)(-\sin t) + (x^2 e^y - 3xy^2)(\cos t) \\ &= -2 \cos t \sin t e^{\sin t} + \sin^4 t + \cos^3 t e^{\sin t} - 3 \cos^2 t \sin^2 t \end{aligned}$$

5. A boat is sailing northeast at 20 km/h. Assuming that the temperature drops at a rate of 0.2°C/km in the northerly direction and 0.3°C/km in the easterly direction, what is the rate of change of temperature with respect to time as observed on the boat?

First we need to parametrize the motion of the boat. Since the boat is sailing northeast, we know that it is travelling along the line $y = x$. Hence we have the parameterization

$\mathbf{r}(t) = \langle \alpha t, \alpha t \rangle$, where α is a constant. Since $|\mathbf{v}(t)| = 20 = \sqrt{2\alpha^2}$, we find that $\alpha = 10\sqrt{2}$ and $\mathbf{r}(t) = \langle x(t), y(t) \rangle = 10\sqrt{2}\langle t, t \rangle$. Now we apply the chain-rule.

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = (-0.3)10\sqrt{2} + (-0.2)10\sqrt{2} = (-5\sqrt{2})^{\circ}\mathrm{C/h}$$

6. Compute the following using implicit differentiation.

(a) $\partial y/\partial z$ if $e^{yz} - x^2 z \ln y = \pi$.

Method 1. Apply the operator $\partial/\partial z$ to both sides of the above equation, considering $y = y(x, z)$ and x fixed.

$$\begin{aligned}
\frac{\partial}{\partial z}\left(e^{yz} - x^2 z \ln y\right) &= \frac{\partial}{\partial z}(\pi) \\
e^{yz}(y\frac{\partial z}{\partial z} + z\frac{\partial y}{\partial z}) - x^2\frac{\partial}{\partial z}(z\ln y) - z\ln y\frac{\partial}{\partial z}(x^2) &= 0 \\
e^{yz}(y + z\frac{\partial y}{\partial z}) - x^2(\frac{z}{y}\frac{\partial y}{\partial z} + \ln y) &= 0 \\
(ze^{yz} - \frac{x^2 z}{y})\frac{\partial y}{\partial z} &= x^2\ln y - ye^{yz} \\
\frac{\partial y}{\partial z} &= \frac{x^2\ln y - ye^{yz}}{ze^{yz} - \frac{x^2 z}{y}}
\end{aligned}$$

Method 2. Let $F(x, y, z) = e^{yz} - x^2 z \ln y - \pi$. We consider $y = y(x, z)$ and x fixed and implicitly differentiate $F(x, y, z) = 0$ with respect to z.

$$\begin{aligned}
\nabla F \cdot \langle \frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z} \rangle &= 0 \\
\langle 2xz\ln y, ze^{yz} - \frac{x^2 z}{y}, ye^{yz} - x^2\ln y \rangle \cdot \langle 0, \frac{\partial y}{\partial z}, 1 \rangle &= 0 \\
(ze^{yz} - \frac{x^2 z}{y})\frac{\partial y}{\partial z} + ye^{yz} - x^2\ln y &= 0 \\
\frac{\partial y}{\partial z} &= \frac{x^2\ln y - ye^{yz}}{ze^{yz} - \frac{x^2 z}{y}}
\end{aligned}$$

(b) dy/dx if $F(x, y, x^2 - y^2) = 0$.

Since we are looking for dy/dx, we are assuming that $y = y(x)$. Hence $z = x^2 - (y(x))^2$. As in method 2 above, we have

$$\begin{aligned}
\nabla F \cdot \langle \frac{dx}{dx}, \frac{dy}{dx}, \frac{dz}{dx} \rangle &= 0 \\
\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \rangle \cdot \langle 1, \frac{dy}{dx}, 2x - 2y\frac{dy}{dx} \rangle &= 0 \\
\frac{\partial F}{\partial x} + 2x\frac{\partial F}{\partial z} + (\frac{\partial F}{\partial y} - 2y\frac{\partial F}{\partial z})\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{-(\frac{\partial F}{\partial x} + 2x\frac{\partial F}{\partial z})}{(\frac{\partial F}{\partial y} - 2y\frac{\partial F}{\partial z})}
\end{aligned}$$

(c) $(\partial y/\partial x)_u$ if $xyuv = 1$ and $x + y + u + v = 0$. [Hint: You should consider $y = y(x, u)$ and $v = v(x, u)$.]

Let $F(x, y, u, v) = xyuv - 1$ and $G(x, y, u, v) = x + y + u + v$. Then we implicitly differentiate $F(x, y, u, v) = 0$ and $G(x, y, u, v) = 0$ with respect to x considering u fixed, $y = y(x, u)$ and $v = v(x, u)$.

$$\begin{aligned} \nabla F \cdot \left\langle \frac{\partial x}{\partial x}, \left(\frac{\partial y}{\partial x}\right)_u, \frac{\partial u}{\partial x}, \left(\frac{\partial v}{\partial x}\right)_u \right\rangle &= 0 \\ \langle yuv, xuv, xyv, xyu \rangle \cdot \left\langle 1, \left(\frac{\partial y}{\partial x}\right)_u, 0, \left(\frac{\partial v}{\partial x}\right)_u \right\rangle &= 0 \\ yuv + xuv \left(\frac{\partial y}{\partial x}\right)_u + xyu \left(\frac{\partial v}{\partial x}\right)_u &= 0 \end{aligned} \tag{1}$$

and

$$\begin{aligned} \nabla G \cdot \left\langle \frac{\partial x}{\partial x}, \left(\frac{\partial y}{\partial x}\right)_u, \frac{\partial u}{\partial x}, \left(\frac{\partial v}{\partial x}\right)_u \right\rangle &= 0 \\ \langle 1, 1, 1, 1 \rangle \cdot \left\langle 1, \left(\frac{\partial y}{\partial x}\right)_u, 0, \left(\frac{\partial v}{\partial x}\right)_u \right\rangle &= 0 \\ 1 + \left(\frac{\partial y}{\partial x}\right)_u + \left(\frac{\partial v}{\partial x}\right)_u &= 0 \end{aligned} \tag{2}$$

Now we have two equations and two unknowns. If we multiply the bottom row of (2) by $-xyu$ and add it to the bottom row of (1). Then we have

$$\begin{aligned} (yuv - xyu) + (xuv - xyu) \left(\frac{\partial y}{\partial x}\right)_u &= 0 \\ \left(\frac{\partial y}{\partial x}\right)_u &= \frac{xyu - yuv}{xuv - xyu} \end{aligned}$$

7. Given a surface $F(x, y, z) = 0$ that defines $x = f(y, z)$, $y = g(x, z)$, and $z = h(x, y)$. Use implicit differentiation to verify that

$$\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial z}\right) = -1.$$

(i) We consider $y = g(x, z)$ and z fixed and implicitly differentiate $F(x, y, z) = 0$ with respect to x.

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = 0 \Leftrightarrow \frac{\partial y}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \tag{3}$$

(ii) We consider $z = h(x, y)$ and x fixed and implicitly differentiate $F(x, y, z) = 0$ with respect to y.

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \Leftrightarrow \frac{\partial z}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \tag{4}$$

(iii) We consider $x = f(y, z)$ and y fixed and implicitly differentiate $F(x, y, z) = 0$ with respect to z.

$$\frac{\partial F}{\partial z} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} = 0 \Leftrightarrow \frac{\partial x}{\partial z} = \frac{-\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial x}} \tag{5}$$

Combining (3), (4), and (5), we see.

$$\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial z}\right) = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\frac{-\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial x}} = -1$$

8. Find second-degree (Taylor) polynomial approximations to the given functions at the points provided.

(a) $f(x, y) = \tan^{-1}(x + xy)$ at the point $(-1, 0)$.

First we need to compute all the first and second order partials. Recall that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, with this in mind the first partials are straightforward.

$$f_x(x, y) = \frac{1 + y}{1 + (x + xy)^2} \text{ and } f_y(x, y) = \frac{x}{1 + (x + xy)^2}$$

Once we have the first partials we just apply the quotient rule to compute the second partials.

$$f_{xx}(x, y) = \frac{-2(1 + y)^2(x + xy)}{(1 + (x + xy)^2)^2}; f_{xy}(x, y) = \frac{1 - (x + xy)^2}{(1 + (x + xy)^2)^2}; f_{yy}(x, y) = \frac{-2x^2(x + xy)}{(1 + (x + xy)^2)^2}.$$

Our quadratic approximation to $f(x, y)$ near the point $(-1, 0)$ is given by, (recall that $\tan^{-1}(-1) = -\pi/4$)

$$\begin{aligned} P_2(x, y) &= f(-1, 0) + f_x(-1, 0)(x + 1) + f_y(-1, 0)y + \frac{1}{2}f_{xx}(-1, 0)(x + 1)^2 \\ &+ f_{xy}(-1, 0)(x + 1)y + \frac{1}{2}f_{yy}(-1, 0)y^2 \\ &= -\frac{\pi}{4} + \frac{1}{2}(x + 1) - \frac{1}{2}y + \frac{1}{4}(x + 1)^2 + \frac{1}{4}y^2 \end{aligned}$$

(b) $g(x, y) = x^3 + 2xy + xy^2$ at the point $(1, 1)$.

The process is the same as in (a) but the computation is simpler. We have $\partial g/\partial x = 3x^2 + 2y + y^2$, $\partial g/\partial y = 2x + 2xy$, $\partial^2 g/\partial x^2 = 6x$, $\partial^2 g/\partial xy = 2 + 2y$, and finally $\partial^2 g/\partial y^2 = 2x$. So we have a quadratic approximation of $g(x, y)$ near $(1, 1)$

$$\begin{aligned} P_2(x, y) &= g(1, 1) + g_x(1, 1)(x - 1) + g_y(1, 1)(y - 1) + \frac{1}{2}g_{xx}(1, 1)(x - 1)^2 \\ &+ g_{yx}(1, 1)(x - 1)(y - 1) + \frac{1}{2}g_{yy}(1, 1)(y - 1)^2 \\ &= 4 + 6(x - 1) + 4(y - 1) + 3(x - 1)^2 + 4(x - 1)(y - 1) + (y - 1)^2 \end{aligned}$$

(c) Use part (b) to estimate $(1.1)^3 + 2(1.1)(.9) + (1.1)(.9)^2$. [Note that the actual value is 4.202.]

Finally an easy question! All we have to do is compute $P_2(1.1, 0.9)$ from part (b). $(1.1)^3 + 2(1.1)(.9) + (1.1)(.9)^2 \approx 4 + 6(0.1) + 4(-0.1) + 3(0.1)^2 + 4(0.1)(-0.1) + (-0.1)^2 = 4 + 0.6 - 0.4 + 0.03 - 0.04 + 0.01 = 4.2$.