M263(2004) Solutions—Assignment 2

1. (a) Differentiating $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$ gives

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}, \qquad \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = 2\mathbf{j}.$$

The desired instantaneous values are $\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{a}(1) = 2\mathbf{j}$.

(b) A normal vector for the desired plane is

$$\mathbf{n} \stackrel{\text{def}}{=} \mathbf{v}(1) \times \mathbf{a}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 2 & 0 \end{vmatrix} = \langle -2,\, 0,\, 2 \rangle .$$

The plane must pass through $\mathbf{r}(1) = (1, 1, 1)$, so its equation is

$$0 = \mathbf{n} \bullet (x-1,\, y-1,\, z-1) = -2(x-1) + 2(z-1).$$

This simplifies to $x = z$.

(c) A vector perpendicular to both $\mathbf{n}$ and $\mathbf{v}$ is

$$\mathbf{n} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \langle -4,\, 4,\, -4 \rangle .$$

So two suitable unit vectors are

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{6}} \langle 1,\, 2,\, 1 \rangle , \qquad \mathbf{w} = \frac{\mathbf{n} \times \mathbf{v}}{|\mathbf{n} \times \mathbf{v}|} = \frac{1}{\sqrt{3}} \langle -1,\, 1,\, -1 \rangle .$$

The signs of $\mathbf{u}$ and/or $\mathbf{w}$ can be reversed without spoiling the required properties.

2. Given $\mathbf{r}(t) = \big(\cos(t),\, \sin(t),\, 2\cos^2(t)\big)$, differentiation gives

$$\begin{aligned} \mathbf{v}(t) &= \dot{\mathbf{r}}(t) = (-\sin(t),\, \cos(t),\, -4\cos(t)\sin(t)) , \\ \mathbf{a}(t) &= \dot{\mathbf{v}}(t) = \big(-\cos(t),\, -\sin(t),\, 4\sin^2(t) - 4\cos^2(t)\big) . \end{aligned}$$

These vectors are perpendicular when

$$\begin{aligned} 0 = \mathbf{v}(t) \bullet \mathbf{a}(t) &= \sin(t)\cos(t) - \sin(t)\cos(t) + 16\sin(t)\cos(t)\left[\cos^2(t) - \sin^2(t)\right] \\ &= 16\sin(t)\cos(t)\,(\cos(t) - \sin(t))\,(\cos(t) + \sin(t)) . \end{aligned}$$

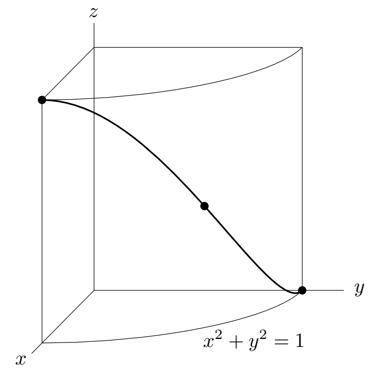
This happens when $\sin(t) = 0$, or $\cos(t) = 0$, or $\sin(t) = \pm\cos(t)$. Among the t-values where $0 \le t < 2\pi$, after which the particle retraces its path, there are 8 solutions:

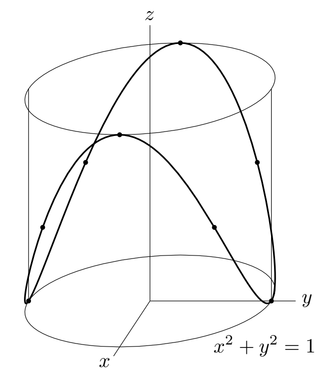
$$t = 0,\ \frac{\pi}{4},\ \frac{2\pi}{4},\ \ldots,\ \frac{7\pi}{4}.$$

The corresponding points on the path are

$$(\pm 1, 0, 2),\ (0, \pm 1, 0),\ \left(\frac{\alpha}{\sqrt{2}}, \frac{\beta}{\sqrt{2}}, 1\right), \quad \text{where} \quad \alpha = \pm 1,\ \beta = \pm 1.$$

Three of these points lie in the first octant, and are shown in the first sketch below. The second sketch shows all eight points.





3. (a) Given that $\mathbf{r}(t) \perp \mathbf{v}(t)$ for all t, we deduce that $2\mathbf{r}(t) \bullet \mathbf{r}'(t) = 0$ for all t. By a calculation done in class, this implies that

$$0 = 2\mathbf{r}(t) \bullet \mathbf{r}'(t) = \frac{d}{dt}\left(\mathbf{r}(t) \bullet \mathbf{r}(t)\right) = \frac{d}{dt}|\mathbf{r}(t)|^2.$$

Since the function $|\mathbf{r}(t)|^2$ has a zero derivative everywhere, it must be constant. Clearly this constant is nonnegative, so call it R^2. Then we have $|\mathbf{r}(t)| = R$ for all t, i.e., the particle is moving on the sphere $x^2 + y^2 + z^2 = R^2$.

(b) Using $R = |\mathbf{r}(0)| = \sqrt{2}$ gives the equation $x^2 + y^2 + z^2 = 2$.

4. A vector normal to this plane is $\mathbf{N} = (3, -2, -1)$. The pebble's acceleration, $\mathbf{a}$, obeys both

$$(1) \quad \mathbf{a} \perp \mathbf{N} \qquad \text{and} \qquad (2) \quad -g\mathbf{k} = \mathbf{a} + t\mathbf{N} \text{ for some } t \in \mathbb{R}.$$

By dotting both sides of (2) with $\mathbf{N}$ and using (1), we find

$$-g\mathbf{k} \bullet \mathbf{N} = 0 + t\mathbf{N} \bullet \mathbf{N}, \quad \text{i.e.,} \quad t = -g\left[\frac{0 + 0 + -1}{9 + 4 + 1}\right] = \frac{g}{14}.$$

Hence, using (2) again, we find

$$\mathbf{a} = -g\mathbf{k} - \frac{g}{14}\mathbf{N} = \frac{g}{14}(-3, 2, -13) . \tag{3}$$

[Alternatively, the component of the gravitational force $-mg\mathbf{k}$ acting perpendicular to the plane is $\mathbf{F}_\perp = \left(\dfrac{-mg\mathbf{k} \bullet \mathbf{N}}{\mathbf{N} \bullet \mathbf{N}}\right)\mathbf{N} = \left(\dfrac{mg}{14}\right)\mathbf{N}$. This component does no work on the pebble: all the work is done by the remaining component, $\mathbf{F}_\parallel$, obtained from $-mg\mathbf{k} = \mathbf{F}_\perp + \mathbf{F}_\parallel$ just as in (3).]

Since $\mathbf{a}$ is constant, $\dot{\mathbf{v}} = \mathbf{a}$ implies $\mathbf{v} = \mathbf{a}t + \mathbf{v}_0$, and $\mathbf{v}_0 = \mathbf{0}$ is given. Next, $\dot{\mathbf{r}} = \mathbf{v} = \mathbf{a}t$ gives $\mathbf{r} = \frac{1}{2}\mathbf{a}t^2 + \mathbf{r}_0$, and $\mathbf{r}_0 = \mathbf{0}$ is given. Thus we have the general formula and particular value

$$\mathbf{r}(t) = \frac{t^2}{2}\mathbf{a} = \frac{gt^2}{28}(-3, 2, -13) ; \qquad \mathbf{r}(2) = \frac{g}{7}(-3, 2, -13) .$$

5. Differentiating $\quad \mathbf{r}(t) = 3(\sin t - t\cos t)\mathbf{i} + 3(\cos t + t\sin t)\mathbf{j} + 2t^2\mathbf{k} \quad$ gives

$$\mathbf{v}(t) = 3t\sin t\,\mathbf{i} + 3t\cos t\,\mathbf{j} + 4t\,\mathbf{k}, \qquad v(t) = \sqrt{9t^2\sin^2 t + 9t^2\cos^2 t + 16t^2} = 5t.$$

(a) The initial point $(0,3,0)$ corresponds to $t=0$; the final point $(-6\pi, 3, 8\pi^2)$ corresponds to $t = 2\pi$. So the arc length between these points is

$$s = \int ds = \int v\,dt = 5\int_{t=0}^{2\pi} t\,dt = 10\pi^2.$$

(b) Using the velocity and speed calculated above gives

$$\widehat{\mathbf{T}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{v} = \frac{3t\sin t\,\mathbf{i} + 3t\cos t\,\mathbf{j} + 4t\,\mathbf{k}}{5t} = \frac{1}{5}\left(3\sin t, 3\cos t, 4\right).$$

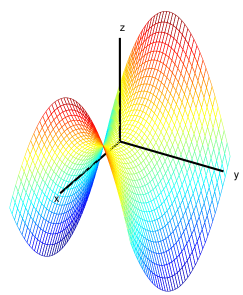
(c) The arc length up to time $t \geq 0$ (assuming $s=0$ when $t=0$) is

$$s(t) = \int_{\theta=0}^{t} v(\theta)\,d\theta = 5\int_0^t \theta\,d\theta = \frac{5}{2}t^2.$$

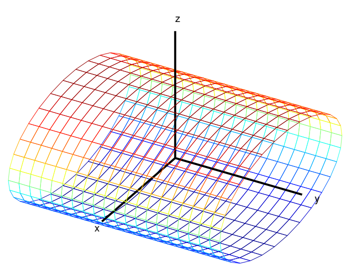
This gives $t = (2s/5)^{1/2}$, and substitution gives the arc-length parametrization

$$\mathbf{r}(s) = 3\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)\mathbf{i} + 3\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)\mathbf{j} + \frac{4s}{5}\mathbf{k}.$$

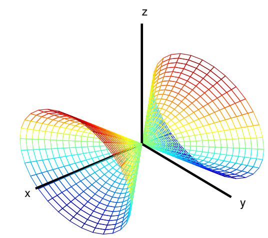
6. (a) This is a hyperbolic paraboloid. In the plane $y=0$, we have $z = x^2$ (an upward-opening parabola), and in the plane $x = 0$, we have $z = -3y^2$ (a downward-opening parabola).



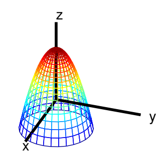
(b) This is an elliptical cylinder. In the plane $y = 0$, the cross-section is $x^2/4 + z^2 = 1$, an ellipse whose major axis occupies the interval $-2 \leq x \leq 2$, and whose minor axis occupies $-1 \leq z \leq 1$.

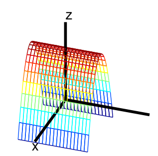


(c) This is a bilateral cone with vertex at the origin. Vertical planes of the form $x =$ const. meet the cone in ellipses parallel to the yz-plane: these ellipses have major axis parallel to the y-axis and minor axis parallel to the z-axis.



7. The graph of f is a rotationally-symmetric paraboloid that opens downward from the vertex at $(0,0,4)$. The graph of g is a parabolic cylinder parallel to the y-axis. Sketches appear below.





8. Use u to parametrize the curve:

$$x = u, \qquad y = u^2, \qquad z = u^3, \qquad u \in \mathbb{R}.$$

Note that $u = 2$ at the point of interest, and that the time-dependence of $\mathbf{r} = (x,y,z)$ comes indirectly through u. Thus, by the chain rule and the product rule,

$$\begin{aligned}
&\dot{x} = \dot{u}, \quad \dot{y} = 2u\dot{u}, \quad \dot{z} = 3u^2\dot{u} &&\Longrightarrow\quad \mathbf{v} = (\dot{x},\, \dot{y},\, \dot{z}) = \left(1,\, 2u,\, 3u^2\right)\dot{u}\\
&\ddot{x} = \ddot{u}, \quad \ddot{y} = 2\dot{u}^2 + 2u\ddot{u}, \quad \ddot{z} = 6u\dot{u}^2 + 3u^2\ddot{u} &&\Longrightarrow\quad \mathbf{a} = (\ddot{x},\, \ddot{y},\, \ddot{z}) = (0,\, 2,\, 6u)\,\dot{u}^2 + \left(1,\, 2u,\, 3u^2\right)\ddot{u}.
\end{aligned}$$

The given phrase, "constant vertical speed $\dot{z} = 3$," implies that

$$3 = \dot{z} = 3u^2\dot{u}, \quad \text{so} \quad 0 = \ddot{z} = 6u\dot{u}^2 + 3u^2\ddot{u}.$$

At the instant of interest, $u = 2$, so these equations become

$$3 = 12\dot{u}, \quad 0 = 12\dot{u}^2 + 12\ddot{u}.$$

The first gives $\dot{u} = 1/4$, and using this in the second gives $\ddot{u} = -1/16$ (instantaneous values at $(2, 4, 8)$). So the instantaneous velocity and acceleration at the point of interest are

$$\mathbf{v} = (1,\, 2u,\, 3u^2)\dot{u} = (1/4,\, 1,\, 3),$$
$$\mathbf{a} = (0,\, 2,\, 12)\left(\frac{1}{16}\right) + (1,\, 4,\, 12)\left(-\frac{1}{16}\right) = \frac{1}{16}\,(-1,\, -2,\, 0)\,.$$

9. A parametric description of the duck's path (with parameter u) is

$$x = 3u, \quad y = 3u^2, \quad z = 2u^3, \qquad u \in \mathbb{R}. \tag{1}$$

The point of interest, $P(3, 3, 2)$, corresponds to the parameter value $u = 1$. The duck's position depends on time indirectly through some functional relation $u = u(t)$ we don't know yet. But the chain rule and product rule give

$$\begin{aligned} \dot{x} &= 3\dot{u}, \quad \dot{y} = 6u\dot{u}, \qquad \dot{z} = 6u^2\dot{u}, \\ \ddot{x} &= 3\ddot{u} \quad \ddot{y} = 6\dot{u}^2 + 6u\ddot{u}, \quad \ddot{z} = 12u\dot{u}^2 + 6u^2\ddot{u}. \end{aligned} \tag{2}$$

Since the duck's speed is constant at 18, the following identity holds for all t:

$$18^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \left[(3)^2 + (6u)^2 + (6u^2)^2\right]\dot{u}^2. \tag{3}$$

In particular, at the instant when $u = 1$,

$$18^2 = [9 + 36 + 36]\,\dot{u}^2 = 9^2\dot{u}^2, \quad \text{i.e.,} \quad \dot{u}^2 = 4 \text{ at } P. \tag{4}$$

Since the duck's x-coordinate is *increasing,* we must have $\dot{u} = 2$ (not $\dot{u} = -2$) at P. Identity (3) holds for all t, so we can differentiate it again:

$$0 = \left[0 + 72u\dot{u} + 144u^3\dot{u}\right]\dot{u}^2 + \left[(3)^2 + (6u)^2 + (6u^2)^2\right](2\dot{u}\ddot{u})\,.$$

At the instant when $u = 1$, we know $\dot{u} = 2$, so at the point P we have

$$0 = [36 + 72]\,(4) + [81]\,(4\ddot{u})\,, \quad \text{i.e.,} \quad \ddot{u} = -\frac{432}{81} = -\frac{16}{3}. \tag{5}$$

Thus, at P, the duck's velocity and acceleration are

$$\mathbf{v} = \langle \dot{x}, \dot{y}, \dot{z}\rangle = \left\langle 3,\, 6u,\, 6u^2\right\rangle \dot{u} = \langle 6, 12, 12\rangle\,,$$
$$\mathbf{a} = \langle \ddot{x}, \ddot{y}, \ddot{z}\rangle = \langle 0,\, 6,\, 12u\rangle\,\dot{u}^2 + \left\langle 3,\, 6u,\, 6u^2\right\rangle \ddot{u} = \langle -16, -8, 16\rangle\,.$$

10. (a) The sphere S has centre $(x_0, y_0, z_0) = (0, 1, -2)$. The distance of this point from the plane $P(k)$ is given by the point-to-plane distance formula:

$$d(k) \stackrel{\text{def}}{=} \frac{|2x_0 + 6y_0 + 3z_0 - k|}{|(2, 6, 3)\,|} = \frac{|-6 + 6 - k|}{\sqrt{4 + 36 + 9}} = \frac{|k|}{7}.$$

Plane $P(k)$ meets S tangentially when $d(k)$ equals the radius of S, which is 3. Enforce this:

$$d(k) = 3 \iff \frac{|k|}{7} = 3 \iff |k| = 21 \iff k = \pm 21.$$

Conclusion: $c = 21$.

(b) Let's use the name $C(k)$ for the circle where S meets $P(k)$, and write $r(k)$ for the radius of $C(k)$. Let $R = 3$ denote the radius of the sphere S. By Pythagoras, $R^2 = d(k)^2 + r(k)^2$. Thus

$$r(k) = \sqrt{R^2 - d(k)^2} = \sqrt{9 - k^2/49} = \frac{1}{7}\sqrt{21^2 - k^2} \qquad (|k| < 21).$$

To find the centre of $C(k)$, we move a distance $d(k)$ in direction parallel to $\mathbf{n} = (2, 6, 3)$ from the centre of S. For $k > 0$ we move forward along $\mathbf{n}$, and for $k < 0$ we move backward. The centre is

$$\mathbf{r}_0 = (0, 1, -2) + \left(\frac{k}{7}\right)\frac{(2, 6, 3)}{|(2, 6, 3)|} = \frac{1}{49}\,(2k, 6k + 49, 3k - 98)\,.$$

(c) The plane of circle $C(k)$ has normal $\mathbf{n} = \langle 2, 6, 3\rangle$, so it contains the vector $\langle -3, 0, 2\rangle$. Another vector it contains, perpendicular to the first, is

$$\mathbf{n} \times \langle -3, 0, 2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 6 & 3 \\ -3 & 0 & 2 \end{vmatrix} = \langle 12, -13, 18\rangle\,.$$

Hence the desired parametrization can be obtained using

$$\begin{aligned} \mathbf{r}_0 &= \frac{1}{49}\,(2k, 6k + 49, 3k - 98) && \text{(the centre of circle } C(k)\text{)}, \\ \mathbf{u} &= \frac{r(k)}{\sqrt{13}}\,\langle -3, 0, 2\rangle && \text{(a vector of length } r(k) \text{ in the given direction)}, \\ \mathbf{w} &= \frac{r(k)}{\sqrt{2077}}\,\langle 12, -13, 18\rangle && \text{(a vector of length } r(k) \text{ in the perpendicular direction)}. \end{aligned}$$

(A different approach may produce a result for $\mathbf{w}$ with the opposite sign. This is also valid.)