MATH 263 ASSIGNMENT 1 SOLUTIONS

1) Find the equation of a sphere if one of its diameters has end points $(2, 1, 4)$ and $(4, 3, 10)$.

Solution. The centre of the sphere is the midpoint of the diameter, which is $\frac{1}{2}\big[(2, 1, 4) + (4, 3, 10)\big] = (3, 2, 7)$. The length of the diameter is $\sqrt{\big|(4, 3, 10) - (2, 1, 4)\big|^2} = \sqrt{2^2 + 2^2 + 6^2} = \sqrt{44}$ so the radius of the sphere is $\frac{1}{2}\sqrt{44} = \sqrt{11}$. The equation of the sphere is $\boxed{(x - 3)^2 + (y - 2)^2 + (z - 7)^2 = 11}$

2) Show that the set of all points P that are twice as far from $(-1, 5, 3)$ as from $(6, 2, -2)$ is a sphere. Find its centre and radius.

Solution. Let the coordinates of a point P be (x, y, z). This point is twice as far from $(-1, 5, 3)$ as from $(6, 2, -2)$ if and only if

$$
\begin{aligned}
&\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} = 2\sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2} \\
\iff &(x+1)^2 + (y-5)^2 + (z-3)^2 = 4(x-6)^2 + 4(y-2)^2 + 4(z+2)^2 \\
\iff &x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 - 6z + 9 = 4x^2 - 48x + 144 + 4y^2 - 16y + 16 + 4z^2 + 16z + 16 \\
\iff &3x^2 - 50x + 3y^2 - 6y + 3z^2 + 22z + 141 = 0 \\
\iff &3\big(x - \tfrac{25}{3}\big)^2 + 3\big(y - 1\big)^2 + 3\big(z + \tfrac{11}{3}\big)^2 + 141 - \tfrac{625}{3} - 3 - \tfrac{121}{3} = 0 \\
\iff &\big(x - \tfrac{25}{3}\big)^2 + \big(y - 1\big)^2 + \big(z + \tfrac{11}{3}\big)^2 = \tfrac{332}{9}
\end{aligned}
$$

This is a circle of $\boxed{\text{centre } \big(\frac{25}{3}, 1, -\frac{11}{3}\big)}$ and $\boxed{\text{radius } \frac{\sqrt{332}}{3}}$.

3) Describe and sketch the set of all points in $\mathbb{R}^3$ that satisfy

a) $x^2 + y^2 + z^2 = 2z$ b) $x^2 + z^2 = 4$ c) $z \geq \sqrt{x^2 + y^2}$

d) $x^2 + y^2 + z^2 = 4,\ z = 1$ e) $x + y + z = 1$

Solution.

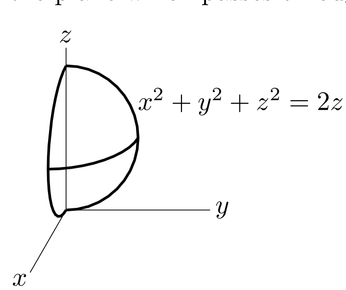
a) Since $x^2 + y^2 + z^2 = 2z$ is equivalent to $x^2 + y^2 + (z-1)^2 = 1$, this is the set of points whose distance from $(0, 0, 1)$ is 1. So this is the sphere of radius 1 centred on $(0, 0, 1)$.

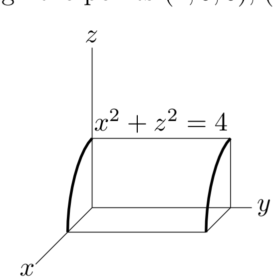
b) For each fixed $y_0 \geq 0$, the curve $x^2 + z^2 = 4,\ y = y_0$ is a circle in the plane $y = y_0$ with centre $(0, y_0, 0)$ and radius 2. As $x^2 + z^2 = 4$ is the union of $x^2 + z^2 = 4,\ y = y_0$ for all possible values of y_0, it is a horizontal stack of vertical circles. The surface is the cylinder of radius 2 centred on the y–axis.

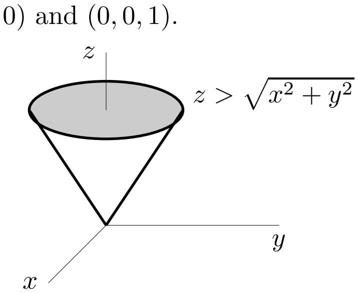
c) For each fixed $z_0 \geq 0$, the curve $z = \sqrt{x^2 + y^2},\ z = z_0$ is a circle in the plane $z = z_0$ with centre $(0, 0, z_0)$ and radius z_0. As $\sqrt{x^2 + y^2} = z$ is the union of $\sqrt{x^2 + y^2} = z,\ z = z_0$ for all possible values of $z_0 \geq 0$, it is a vertical stack of horizontal circles whose radii increase linearly with z. It is a cone centered on the z–axis. $z > \sqrt{x^2 + y^2}$ is the region above this cone. It is a solid cone.

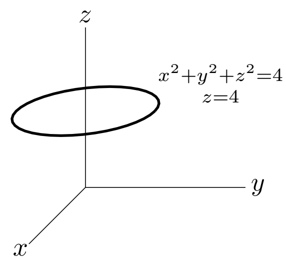
d) This is the circle of radius $\sqrt{3}$ centred on $(0, 0, 1)$ that lies parallel to the xy–plane.

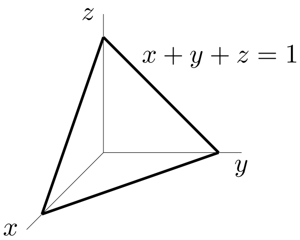
e) This is the plane which passes through the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.





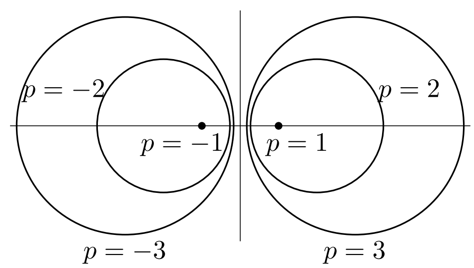






4) The pressure $p(x,y)$ at the point (x,y) is determined by $x^2-2px+y^2+1=0$. Sketch the isobars (curves of constant pressure).

Solution. The isobar for pressure p is the curve $x^2-2px+y^2+1=0$, or equivalently, $(x-p)^2+y^2=p^2-1$. For $|p|>1$ this is the circle of radius $\sqrt{p^2-1}$ centred on $(p,0)$. For $|p|=1$ it is the point $(p,0)$ and for $|p|<1$, no real (x,y) satisfies the equation. Here is a sketch showing six typical isobars.



5) Compute the dot product of the vectors $\vec{a}$ and $\vec{b}$. Find the angle between them.

a) $\vec{a}=\langle -1,1\rangle$, $\vec{b}=\langle 1,1\rangle$ b) $\vec{a}=\langle 1,1\rangle$, $\vec{b}=\langle 2,2\rangle$

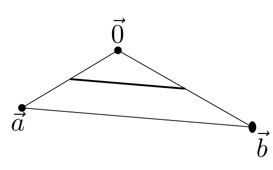
c) $\vec{a}=\langle 1,2,1\rangle$, $\vec{b}=\langle -1,1,1\rangle$

Solution.

$$\begin{array}{llll}
\text{a)} & \vec{a}\cdot\vec{b}=\langle -1,1\rangle\cdot\langle 1,1\rangle=0 & \cos\theta=\dfrac{0}{\sqrt{2}\sqrt{2}}=0 & \theta=90^\circ \\
\text{b)} & \vec{a}\cdot\vec{b}=\langle 1,1\rangle\cdot\langle 2,2\rangle=4 & \cos\theta=\dfrac{4}{\sqrt{2}\sqrt{8}}=1 & \theta=0^\circ \\
\text{c)} & \vec{a}\cdot\vec{b}=\langle 1,2,1\rangle\cdot\langle -1,1,1\rangle=2 & \cos\theta=\dfrac{2}{\sqrt{6}\sqrt{3}}=.4714 & \theta=61.87^\circ
\end{array}$$

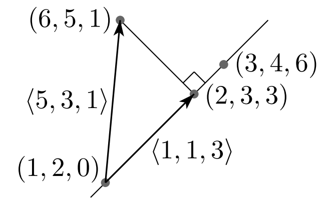
6) Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

Solution. Pick any two sides to the triangle. We can always put the vertex of the triangle joining the two sides at the origin. Call the other two vertices $\vec{a}$ and $\vec{b}$. The midpoint of the side joining the vertex $\vec{0}$ to the vertex $\vec{a}$ is $\frac{1}{2}\vec{a}$. The midpoint of the side joining the vertex $\vec{0}$ to the vertex $\vec{b}$ is $\frac{1}{2}\vec{b}$. The vector joining the two midpoints is $\frac{1}{2}\vec{b}-\frac{1}{2}\vec{a}$. This is indeed parallel to the third side, which joins $\vec{a}$ and $\vec{b}$, and half its length.



7) Drop a perpendicular from the point $(6,5,1)$ to the line, L, through the points $(1,2,0)$ and $(3,4,6)$. Where does the perpendicular hit L?

Solution. The vector from $(1,2,0)$ to $(6,5,1)$ is $\langle 6,5,1\rangle - \langle 1,2,0\rangle = \langle 5,3,1\rangle$. The vector $\langle 3,4,6\rangle - \langle 1,2,0\rangle = \langle 2,2,6\rangle$ lies in the line. The projection of $\langle 5,3,1\rangle$ on $\langle 2,2,6\rangle$ is



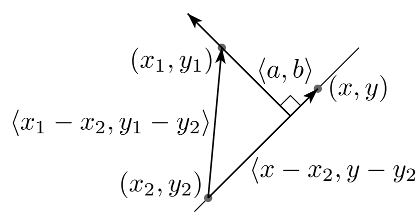
$$\frac{\langle 5,3,1\rangle\cdot\langle 2,2,6\rangle}{|\langle 2,2,6\rangle|^2}\,\langle 2,2,6\rangle = \frac{22}{44}\,\langle 2,2,6\rangle = \langle 1,1,3\rangle$$

The perpendicular hits L at $(1,2,0)+(1,1,3) = \boxed{(2,3,3)}$.

8) Use a projection to derive a formula for the distance from a point (x_1,y_1) to the line $ax+by=c$. Here, a and b are not both zero.

Solution. Let (x_2,y_2) be any point on the line. Then $ax_2+by_2=c$. If (x,y) is any other point on the line, then $ax+by=c$ so that $a(x_2-x)+b(y_2-y)=c-c=0$. That is, $\langle a,b\rangle$ is perpendicular to $\langle x_2-x, y_2-y\rangle$. As $\langle x_2-x, y_2-y\rangle$ is an arbitrary vector lying on the line, $\langle a,b\rangle$ is a normal to the line. The distance from (x_1,y_1) to $ax+by=c$ is the length of the projection of the vector $\langle x_1-x_2, y_1-y_2\rangle$ on the vector $\langle a,b\rangle$, which is

$$\frac{|\langle x_1-x_2, y_1-y_2\rangle\cdot\langle a,b\rangle|}{|\langle a,b\rangle|} = \frac{|ax_1-ax_2+by_1-by_2|}{\sqrt{a^2+b^2}} = \boxed{\frac{|ax_1+by_1-c|}{\sqrt{a^2+b^2}}}$$



9) Compute $\langle 1,2,3\rangle \times \langle 4,5,6\rangle$.

Solution.

$$\langle 1,2,3\rangle\times\langle 4,5,6\rangle = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \hat{\imath}(2\times 6-3\times 5)-\hat{\jmath}(1\times 6-3\times 4)+\hat{\mathbf{k}}(1\times 5-2\times 4) = \boxed{-3\hat{\imath}+6\hat{\jmath}-3\hat{\mathbf{k}}}$$

10) Prove that

a) $\hat{\imath}\times\hat{\jmath} = \hat{\mathbf{k}}$

b) $\vec{a}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{a}\times\vec{b}) = 0$

c) $|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

Solution. a)

$$\hat{\imath}\times\hat{\jmath} = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} = \hat{\imath}(0\times 0-0\times 1)-\hat{\jmath}(1\times 0-0\times 0)+\hat{\mathbf{k}}(1\times 1-0\times 0) = \boxed{\hat{\mathbf{k}}}$$

b)

$$\begin{aligned}\vec{a}\cdot(\vec{a}\times\vec{b}) &= a_1\big(a_2b_3-a_3b_2\big)-a_2\big(a_1b_3-a_3b_1\big)+a_3\big(a_1b_2-a_2b_1\big)=0\\ \vec{b}\cdot(\vec{a}\times\vec{b}) &= b_1\big(a_2b_3-a_3b_2\big)-b_2\big(a_1b_3-a_3b_1\big)+b_3\big(a_1b_2-a_2b_1\big)=0\end{aligned}$$

c) Just compare

$$\begin{aligned}|\vec{a}\times\vec{b}|^2 &= (a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2\\ &= a_2^2b_3^2-2a_2b_3a_3b_2+a_3^2b_2^2+a_3^2b_1^2-2a_3b_1a_1b_3+a_1^2b_3^2+a_1^2b_2^2-2a_1b_2a_2b_1+a_2^2b_1^2\end{aligned}$$

and

$$\begin{aligned}|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2 &= \big(a_1^2+a_2^2+a_3^2\big)\big(b_1^2+b_2^2+b_3^2\big)-\big(a_1b_1+a_2b_2+a_3b_3\big)^2\\ &= a_1^2b_2^2+a_1^2b_3^2+a_2^2b_1^2+a_2^2b_3^2+a_3^2b_1^2+a_3^2b_2^2-\big(2a_1b_1a_2b_2+2a_1b_1a_3b_3+2a_2b_2a_3b_3\big)\end{aligned}$$

11) Find the equation of the sphere which has the two planes $x+y+z=3,\ x+y+z=9$ as tangent planes if the centre of the sphere is on the planes $2x-y=0,\ 3x-z=0$.

Solution. The planes $x+y+z=3$ and $x+y+z=9$ are parallel. So the centre lies on $x+y+z=6$ (the plane midway between $x+y+z=3$ and $x+y+z=9$) as well as on $y=2x$ and $z=3x$. Solving,

$$y=2x,\ z=3x,\ x+y+z=6 \Rightarrow x+2x+3x=6 \Rightarrow x=1,\ y=2,\ z=3$$

So the centre is at $(1,2,3)$. The normal to $x+y+z=3$ is $\langle 1,1,1\rangle$. The points $(1,1,1)$ on $x+y+z=3$ and $(3,3,3)$ on $x+y+z=9$ differ by a vector, $\langle 2,2,2\rangle$, which is a multiple of this normal. So the distance between the planes is $|\langle 2,2,2\rangle|=2\sqrt{3}$ and the radius of the sphere is $\sqrt{3}$. The sphere is

$$\boxed{(x-1)^2+(y-2)^2+(z-3)^2=3}$$

12) Find the equation of the plane that passes through the point $(-2,0,-1)$ and through the line of intersection of $2x+3y-z=0,\ x-4y+2z=-5$.

Solution. First we'll find two points on the line of intersection of $2x+3y-z=0,\ x-4y+2z=-5$. This will give us three points on the plane.

$$\left\{\begin{array}{l}2x+3y-z=\ \ 0\\ x-4y+2z=-5\end{array}\right\} \iff \left\{\begin{array}{l}2x+3y=z\\ \ \ x-4y=-2z-5\end{array}\right\} \iff \left\{\begin{array}{l}2x+3y=z\\ \qquad 11y=5(z+2)\end{array}\right\}$$

In the last step, we subtracted twice the second equation from the first. So if $z=-2$, then $y=0$ and $x=-1$. And if $z=-\frac{15}{2}$, then $y=-\frac{5}{2}$ and $x=0$. So we conclude that the three points $(-2,0,-1)$, $(-1,0,-2)$ and $(0,-\frac{5}{2},-\frac{15}{2})$ must all lie on the plane. So the two vectors $\langle -2,0,-1\rangle-\langle -1,0,-2\rangle=\langle -1,0,1\rangle$ and $\left\langle 0,-\frac{5}{2},-\frac{15}{2}\right\rangle-\langle -1,0,-2\rangle=\left\langle 1,-\frac{5}{2},-\frac{11}{2}\right\rangle$ must be parallel to the plane. So the normal to the plane is $\langle -1,0,1\rangle\times\left\langle 1,-\frac{5}{2},-\frac{11}{2}\right\rangle=\left\langle \frac{5}{2},-\frac{9}{2},\frac{5}{2}\right\rangle$ or, equivalently $\vec{n}=\langle 5,-9,5\rangle$. The equation of the plane is

$$5(x+2)-9y+5(z+1)=0 \text{ or } \boxed{5x-9y+5z=-15}$$

13) Find the equations of the line through $(2,-1,-1)$ and parallel to each of the two planes $x+y=0$ and $x-y+2z=0$. Express the equations of the line in vector and scalar parametric forms and in symmetric form.

Solution. One vector normal to $x+y=0$ is $\langle 1,1,0\rangle$. One vector normal to $x-y+2z=0$ is $\langle 1,-1,2\rangle$. The vector $\langle 1,-1,-1\rangle$ is perpendicular to both of those normals and hence is parallel to both planes. So $\langle 1,-1,-1\rangle$ is also parallel to the line. The vector parametric equation of the line is

$$\boxed{\vec{x}=(2,-1,-1)+t(1,-1,-1)}$$

The scalar parametric equations of the line are

$$\boxed{x=2+t,\ y=-1-t,\ z=-1-t}$$

The symmetric equations are

$$t=\boxed{x-2=-y-1=-z-1}$$