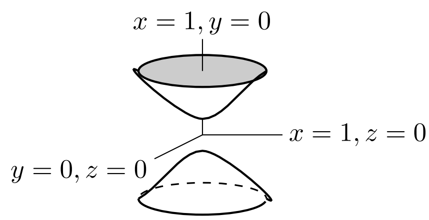
Math 263 Midterm 1 Solutions

1a) The points $(1,1,0)$ and $(0,0,2)$ are on the line $2x+z=2, 2y+z=2$ and hence on $\mathcal{P}$. So the vector from $(2,0,0)$ to $(1,1,0)$, which is $\langle -1,1,0\rangle$, is parallel to $\mathcal{P}$. So is the vector from $(2,0,0)$ to $(0,0,2)$, which is $\langle -2,0,2\rangle$. So the vector $\langle -1,1,0\rangle \times \langle -2,0,2\rangle = \langle 2,2,2\rangle$ is a normal vector for $\mathcal{P}$. So is the vector $\frac{1}{2}\langle 2,2,2\rangle = \langle 1,1,1\rangle$. The equation of $\mathcal{P}$ is $\langle 1,1,1\rangle \cdot \big(\langle x,y,z\rangle - \langle 2,0,0\rangle\big) = 0$ or $\boxed{x+y+z=2}$

b) The direction vector for $\mathcal{L}$, namely $\langle 1,0,-1\rangle$, is perpendicular to $\langle 1,1,1\rangle$ which in turn is perpendicular to $\mathcal{P}$. So $\langle 1,0,-1\rangle$, and hence $\mathcal{L}$, is parallel to $\mathcal{P}$.

c) The point $(-2,-1,-1)$ is on $\mathcal{L}$. The point $(0,1,1)$ is on $\mathcal{P}$. The vector joining them, namely $\langle 2,2,2\rangle$ is normal to $\mathcal{P}$. So the distance between $\mathcal{L}$ and $\mathcal{P}$ is $|\langle 2,2,2\rangle| = 2\sqrt{3}$.

2a) Completing the square, we may rewrite the equation of $\mathcal{S}$ as $(x-1)^2+y^2=z^2-1$. The part of the surface in the plane $z=z_0$ is $(x-1)^2+y^2=z_0^2-1$. If $|z_0|<1$ then $z_0^2-1<0$ and no x,y satisfies the equation. Otherwise, $(x-1)^2+y^2=z^2-1,\ z=z_0$ is a circle which is parallel to the xy–plane, is centred on $x=1$, $y=0$ and has radius $\sqrt{z_0^2-1}$. This radius is zero when $z_0=\pm 1$ and increases as z_0 out from -1 or 1. For large z_0, the radius is approximately $|z_0|$. So our surface consists of a bunch of horizontal circles stacked vertically, starting with circles of radius zero at $z=-1,1$ and growing in size as we move out. The sketch is



b) A normal vector to $\mathcal{S}$ at $(0,1,\sqrt{3})$ is

$$\vec{\nabla}(x^2-2x+y^2-z^2+2)\big|_{(0,1,\sqrt{3})} = \langle 2x-2, 2y, -2z\rangle\big|_{(0,1,\sqrt{3})} = \big\langle -2, 2, -2\sqrt{3}\big\rangle$$

An equation of the tangent plane is

$$\big\langle 1,-1,\sqrt{3}\big\rangle \cdot \big(\langle x,y,z\rangle - \big\langle 0,1,\sqrt{3}\big\rangle\big) = 0 \text{ or } \boxed{x-y+\sqrt{3}z=2}$$

c) The curve of intersection is $(x-1)^2+y^2=z^2-1,\ z=\sqrt{5}$ or $(x-1)^2+y^2=4,\ z=\sqrt{5}$ which is a horizontal circle of radius 2 centred on $(1,0,\sqrt{5})$. One parametrization is

$$\langle x,y,z\rangle = \big\langle 1,0,\sqrt{5}\big\rangle + 2\cos t\,\hat{\imath} + 2\sin t\,\hat{\jmath} \qquad 0 \le t \le 2\pi$$

3a) If $x+y\neq 2$,

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}\frac{x^2-2x+y^2+y-1}{x+y-2} = \frac{2y+1}{x+y-2} - \frac{x^2-2x+y^2+y-1}{(x+y-2)^2}$$

In particular,

$$\frac{\partial f}{\partial y}(1,2) = \frac{2\times 2+1}{1+2-2} - \frac{1-2\times 1+2^2+2-1}{(1+2-2)^2} = \boxed{1}$$

b) By definition

$$\begin{aligned}\frac{\partial f}{\partial y}(1,1) &= \lim_{\Delta y\to 0}\frac{f(1,1+\Delta y)-f(1,1)}{\Delta y} = \lim_{\Delta y\to 0}\frac{1}{\Delta y}\Big[\frac{1-2+(1+\Delta y)^2+(1+\Delta y)-1}{1+(1+\Delta y)-2} - 3\Big] \\ &= \lim_{\Delta y\to 0}\frac{1}{\Delta y}\Big[\frac{(\Delta y)^2+3\Delta y}{\Delta y} - 3\Big] = \lim_{\Delta y\to 0} 1 = \boxed{1}\end{aligned}$$

Warnings:

a) Knowing the value of a function at one point tells you nothing about the value of any derivative of that function at that point. In particular, knowing that $f(1,1)=3$ does not necessarily mean that f is a constant, so that $f_y(1,1)=0$. For example, the function $f(x,y)=3+7(y-1)$ obeys $f(1,1)=3$, but $f_y(1,1)=7$.

b) $f_y(1,1)$ is the limit of $f_y(x,y)$ as (x,y) approaches $(1,1)$ **only** if $f_y(x,y)$ is continuous at $(1,1)$. In this question, $f_y(x,y)$ is not continuous at $(1,1)$ and cannot be computed as the limit of $\frac{2y+1}{x+y-2} - \frac{x^2-2x+y^2+y-1}{(x+y-2)^2}$ as (x,y) approaches $(1,1)$.

4a) The temperature gradient at P is

$$\vec{\nabla}T(2,-1,3) = \big\langle yz\cos(xyz+6), xz\cos(xyz+6), xy\cos(xyz+6)\big\rangle\Big|_{(2,-1,3)} =< -3,6,-2 >$$

The direction of travel is $\langle 1-2, 1+1, 1-3\rangle = \langle -1,2,-2\rangle$. A unit vector in this direction is $\frac{1}{3}\langle -1,2,-2\rangle$. So the directional derivative, i.e. the rate of change per unit distance, is $< -3,6,-2 > \cdot \frac{1}{3}\langle -1,2,-2\rangle = \boxed{\frac{19}{3}^\circ/\mathrm{m}}$

b)

$$\begin{aligned} T(2.1,-0.95,3.25) &\approx T(2,-1,3) + 0.1T_x(2,-1,3) + 0.05T_y(2,-1,3) + 0.25T_z(2,-1,3) \\ &\approx 20 + 0.1(-3) + 0.05\times 6 + 0.25(-2) \\ &= \boxed{19.5^\circ} \end{aligned}$$

c) The bee should move in the direction of the temperature gradient, which is $\boxed{< -3,6,-2 >}$.