Midterm #3 Formulas

LINE INTEGRALS

Along parametrized curve $\mathcal{C}$: $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$, $a \le t \le b$, $\quad \int_{\mathcal{C}} f(\vec{\mathbf{r}})\, ds = \int_a^b f(\vec{\mathbf{r}}(t)) \left| \frac{d\vec{\mathbf{r}}}{dt} \right| dt, \quad W = \int_{\mathcal{C}} dW = \int_{\mathcal{C}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \bullet \frac{d\vec{\mathbf{r}}}{dt} dt$

Necessary condition for $\vec{\mathbf{F}}(x,y,z) = F_1(x,y,z)\,\vec{\mathbf{\imath}} + F_2(x,y,z)\,\vec{\mathbf{\jmath}} + F_3(x,y,z)\,\vec{\mathbf{k}}$ conservative:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \quad \text{and} \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \quad \text{and} \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

SURFACE NORMALS AND AREA ELEMENTS

Parametric surface $\vec{\mathbf{r}} = \vec{\mathbf{r}}(u,v)$: $\quad \vec{\mathbf{n}} = \pm\left(\frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}\right) \quad d\vec{\mathbf{S}} = \pm\left(\frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}\right) du\,dv \quad dS = \left|\frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}\right| du\,dv$

Surface with equation $z = g(x,y)$: $\quad \vec{\mathbf{n}} = \pm\left(-g_1\,\vec{\mathbf{\imath}} - g_2\,\vec{\mathbf{\jmath}} + \vec{\mathbf{k}}\right) \quad d\vec{\mathbf{S}} = \pm\left(-g_1\,\vec{\mathbf{\imath}} - g_2\,\vec{\mathbf{\jmath}} + \vec{\mathbf{k}}\right) dx\,dy \quad dS = \sqrt{(g_1)^2 + (g_2)^2 + 1}\; dx\,dy$

Level surface $G(x,y,z) = 0$: $\quad \vec{\mathbf{n}} = \pm\nabla G(x,y,z) \quad d\vec{\mathbf{S}} = \pm\frac{\nabla G(x,y,z)}{|\partial G/\partial z|} dx\,dy \quad dS = \left|\frac{\nabla G(x,y,z)}{\partial G/\partial z}\right| dx\,dy$

Other projections: $\quad d\vec{\mathbf{S}} = \frac{\vec{\mathbf{n}}}{|\vec{\mathbf{n}} \bullet \vec{\mathbf{k}}|} dx\,dy = \frac{\vec{\mathbf{n}}}{|\vec{\mathbf{n}} \bullet \vec{\mathbf{\jmath}}|} dx\,dz = \frac{\vec{\mathbf{n}}}{|\vec{\mathbf{n}} \bullet \vec{\mathbf{\imath}}|} dy\,dz \quad dS = |d\vec{\mathbf{S}}|$

POLAR COORDINATES

Transformation: $\quad x = r\cos\theta, \quad y = r\sin\theta \quad\quad r = \sqrt{x^2+y^2}, \quad \tan\theta = y/x$

Area element: $\quad dA = r\,dr\,d\theta$

CYLINDRICAL COORDINATES

Transformation: $\quad x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \quad\quad r = \sqrt{x^2+y^2}, \quad \tan\theta = y/x, \quad z = z$

Volume element: $\quad dV = r\,dr\,d\theta\,dz \quad\quad$ Surface area element (on $r = a$): $\quad dS = a\,d\theta\,dz$

SPHERICAL COORDINATES

Transformation: $\quad x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}, \quad \tan\phi = \frac{r}{z} = \frac{\sqrt{x^2+y^2}}{z}, \quad \tan\theta = \frac{y}{x}$$

Volume element: $\quad dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \quad\quad$ Surface area element (on $\rho = a$): $\quad dS = a^2\sin\phi\,d\phi\,d\theta$

TRIGONOMETRIC IDENTITIES

$\sin^2 x + \cos^2 x = 1 \quad\quad \sin(-x) = -\sin x \quad\quad \cos(-x) = \cos x$

$\sec^2 x = 1 + \tan^2 x \quad\quad \sin(\pi - x) = \sin x \quad\quad \cos(\pi - x) = -\cos x$

$\csc^2 x = 1 + \cot^2 x \quad\quad \sin\left(\frac{\pi}{2} - x\right) = \cos x \quad\quad \cos\left(\frac{\pi}{2} - x\right) = \sin x$

$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y \quad\quad \cos(x \pm y) = \cos x\cos y \mp \sin x\sin y \quad\quad \tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$

$\sin 2x = 2\sin x\cos x \quad\quad \sin^2 x = \frac{1 - \cos 2x}{2} \quad\quad \cos^2 x = \frac{1 + \cos 2x}{2}$

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x$

INTEGRALS

$\int \sec^2 x\,dx = \tan x + C \quad\quad \int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C \quad\quad \int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

$\int \tan x\,dx = \ln|\sec x| + C \quad\quad \int \sin^3 x\,dx = \frac{1}{3}\cos^3 x - \cos x \quad\quad \int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x$

$\int \tan^2 x\,dx = \tan x - x + C \quad\quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \quad (a > 0) \quad\quad \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C \quad (a > 0)$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C \quad (a > 0,\ |x| < a)$

Adapted from R. A. Adams, *Calculus, Several Variables,* Addison-Wesley, 2003.