Math 263 Assignment 9

Due December 1

1) Let $\vec{F} = (x - yz)\hat{\imath} + (y + xz)\hat{\jmath} + (z + 2xy)\hat{k}$ and let
   S_1 be the portion of the cylinder $x^2 + y^2 = 2$ that lies inside the sphere $x^2 + y^2 + z^2 = 4$
   S_2 be the portion of $x^2 + y^2 + z^2 = 4$ that lies outside the cylinder $x^2 + y^2 = 2$
   V be the volume bounded by S_1 and S_2

   Compute
   a) $\int\int_{S_1} \vec{F} \cdot \hat{n}\, dS$ with $\hat{n}$ pointing inward
   b) $\int\int\int_V \vec{\nabla} \cdot F\, dV$
   c) $\int\int_{S_2} \vec{F} \cdot \hat{n}\, dS$ with $\hat{n}$ pointing outward

   Use the divergence theorem to answer at least one of parts (a), (b) and (c).

2) Evaluate the integral $\iint_S \vec{F} \cdot \hat{n}\, dS$, where $\vec{F} = (x, y, 1)$ and S is the surface $z = 1 - x^2 - y^2$, for $x^2 + y^2 \le 1$, by two methods.
   a) First, by direct computation of the surface integral.
   b) Second, by using the divergence theorem.

3a) By applying the divergence theorem to $\vec{F} = \phi\vec{a}$, where $\vec{a}$ is an arbitrary constant vector, show that

$$\iiint_V \vec{\nabla}\phi\, dV = \iint_{\partial V} \phi\hat{\mathbf{n}}\, dS$$

b) Show that the centroid $(\bar{x}, \bar{y}, \bar{z})$ of a solid V is given by

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{2\,\mathrm{vol}(V)} \iint_{\partial V} (x^2 + y^2 + z^2)\, \hat{\mathbf{n}}\, dS$$

4) Find the flux of $\vec{F} = (y + xz)\hat{\boldsymbol{\imath}} + (y + yz)\hat{\boldsymbol{\jmath}} - (2x + z^2)\hat{\mathbf{k}}$ upward through the first octant part of the sphere $x^2 + y^2 + z^2 = a^2$.

5) Let $\vec{E}(\vec{r})$ be the electric field due to a charge configuration that has density $\rho(\vec{r})$. Gauss' law states that, if V is any solid in $\mathbb{R}^3$ with surface ∂V, then the electric flux

$$\iint_{\partial V} \vec{E} \cdot \hat{\mathbf{n}}\, dS = 4\pi Q \qquad \text{where} \qquad Q = \iiint_V \rho\; dV$$

is the total charge in V. Here, as usual, $\hat{\mathbf{n}}$ is the outward pointing unit normal to ∂V. Show that

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = 4\pi\rho(\vec{r})$$

for all $\vec{r}$ in $\mathbb{R}^3$. This is one of Maxwell's equations. Assume that $\vec{\nabla} \cdot \vec{E}(\vec{r})$ and $\rho(\vec{r})$ are well–defined and continuous everywhere.

6) Evaluate, both by direct integration and by Stokes' Theorem, $\oint_C (z\, dx + x\, dy + y\, dz)$ where C is the circle $x + y + z = 0$, $x^2 + y^2 + z^2 = 1$. Orient C so that its projection on the xy–plane is counterclockwise.

7) Evaluate $\oint_C (x \sin y^2 - y^2)dx + (x^2 y \cos y^2 + 3x)dy$ where C is the counterclockwise boundary of the trapezoid with vertices $(0, -2)$, $(1, -1)$, $(1, 1)$ and $(0, 2)$.

8) Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = ye^x\hat{\boldsymbol{\imath}} + (x + e^x)\hat{\boldsymbol{\jmath}} + z^2\hat{\mathbf{k}}$ and C is the curve

$$\vec{r}(t) = (1 + \cos t)\hat{\boldsymbol{\imath}} + (1 + \sin t)\hat{\boldsymbol{\jmath}} + (1 - \sin t - \cos t)\hat{\mathbf{k}}$$

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9) Let C be the intersection of $x+2y-z=7$ and $x^2-2x+4y^2=15$. The curve C is oriented counterclockwise when viewed from high on the z–axis. Let

$$\vec{F} = \big(x^3e^{-x} + yz\big)\hat{\boldsymbol{\imath}} + \big(\tfrac{\sin y}{y} + \sin z - x^2\big)\hat{\boldsymbol{\jmath}} + \big(xy + y\cos z\big)\hat{\mathbf{k}}$$

Evaluate $\oint_C \vec{F}\cdot d\vec{r}$.

10) Consider $\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{\mathbf{n}}\,dS$ where S is the portion of the sphere $x^2+y^2+z^2=1$ that obeys $x+y+z\ge 1$, $\hat{\mathbf{n}}$ is the upward pointing normal to the sphere and $\vec{F} = (y-z)\hat{\boldsymbol{\imath}} + (z-x)\hat{\boldsymbol{\jmath}} + (x-y)\hat{\mathbf{k}}$. Find another surface S' with the property that $\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{\mathbf{n}}\,dS = \iint_{S'} (\vec{\nabla}\times\vec{F})\cdot\hat{\mathbf{n}}\,dS$ and evaluate $\iint_{S'} (\vec{\nabla}\times\vec{F})\cdot\hat{\mathbf{n}}\,dS$.

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