Math 263 Assignment #8

Due in class on Wednesday 24 November 2004

1. For each of the following vectors fields $\vec{\mathbf{F}}$, find div $\vec{\mathbf{F}}$ and curl $\vec{\mathbf{F}}$.

$$\text{(a) } \vec{\mathbf{F}}(x,y,z) = z\vec{\mathbf{j}} - y\vec{\mathbf{k}} \quad \text{(b) } \vec{\mathbf{F}}(x,y,z) = \langle x^2, y, z\rangle \quad \text{(c) } \vec{\mathbf{F}}(x,y,z) = \langle x+y, -y^2, -2z\rangle$$

2. For each of the following oriented surfaces $\mathcal{S}$, (i) sketch $\mathcal{S}$, (ii) parametrize $\mathcal{S}$, (iii) find the vector and scalar area elements $d\vec{\mathbf{S}}$ and dS for your parametrization, (iv) calculate the indicated surface or flux integral.

   (a) $\mathcal{S}$ given by $z = x^2y^2$, $-1 \le x \le 1$, $-1 \le y \le 1$ oriented positive side upward. Calculate $\iint_{\mathcal{S}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}}$ for $\vec{\mathbf{F}} = x\vec{\mathbf{\imath}} + \vec{\mathbf{\jmath}} + z\vec{\mathbf{k}}$.

   (b) $\mathcal{S}$ surface of ellipsoid $4x^2 + 4y^2 + z^2 - 6z + 5 = 0$ oriented inward. Calculate surface area of $\mathcal{S}$.

   (c) $\mathcal{S}$ surface of intersection of sphere $x^2 + y^2 + z^2 \le 4$ and plane $z = 1$ oriented away from the origin. Calculate flux away from the origin of the electrical field $\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \dfrac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|^3}$.

3. Let $\mathcal{S}$ be the portion of the surface $x^2 + 1 = y^2 + z^2$ bounded by the planes $x = -1$, $x = 2$, and lying above the xy-plane. Calculate the surface integral $\iint_{\mathcal{S}} z\sqrt{\dfrac{1+2x^2}{y^2+z^2}}\, dS$.

   [Hint: You may find it helpful to restate the problem, exchanging the variables x, y, and z throughout to make the surface symmetric around the z-axis.]

4. Use geometric reasoning to find $I = \iint_{\mathcal{S}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}}$ "by inspection" in the three situations below. Briefly explain your answers. (In all parts, a and b are positive constants.)

   (a) $\vec{\mathbf{F}}(x,y,z) = x\vec{\mathbf{\imath}} + y\vec{\mathbf{\jmath}} + z\vec{\mathbf{k}}$, and $\mathcal{S}$ is the surface consisting of three squares with one corner at the origin and positive sides facing the first octant. The squares have sides $b\vec{\mathbf{\imath}}$ and $b\vec{\mathbf{\jmath}}$, $b\vec{\mathbf{\jmath}}$ and $b\vec{\mathbf{k}}$, and $b\vec{\mathbf{\imath}}$ and $b\vec{\mathbf{k}}$, respectively.

   (b) $\vec{\mathbf{F}}(x,y,z) = (x\vec{\mathbf{\imath}} + y\vec{\mathbf{\jmath}})\ln(x^2+y^2)$, and $\mathcal{S}$ is the surface of the cylinder (including the top and bottom) where $x^2 + y^2 \le a^2$ and $0 \le z \le b$.

   (c) $\vec{\mathbf{F}}(x,y,z) = (x\vec{\mathbf{\imath}} + y\vec{\mathbf{\jmath}} + z\vec{\mathbf{k}})e^{-(x^2+y^2+z^2)}$, and $\mathcal{S}$ is the spherical surface $x^2 + y^2 + z^2 = a^2$.

5. Let $\mathcal{S}$ be the boundary surface for the solid given by $0 \le z \le \sqrt{4-y^2}$ and $0 \le x \le \frac{\pi}{2}$.

   (a) Find the outward unit normal vector field $\hat{\mathbf{N}}$ on each of the four sides of $\mathcal{S}$.

   (b) Find the total outward flux of $\vec{\mathbf{F}} = 4\sin(x)\,\vec{\mathbf{\imath}} + z^3\,\vec{\mathbf{\jmath}} + yz^2\,\vec{\mathbf{k}}$ through $\mathcal{S}$.

   Do the calculations directly—don't use the Divergence Theorem. [Hint: Flux integrals for three of the four sides can be calculated geometrically.]

6. Simplify the following expressions for smooth vector fields $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ and smooth scalar fields ϕ and ψ. [Hint: You may find Theorem 3 on pages 954–955 helpful.]

   (a) $\nabla \bullet (\nabla\phi \times \nabla\psi)$

   (b) $\nabla \bullet (\phi\vec{\mathbf{F}} + \vec{\mathbf{G}}) - (\nabla\phi) \bullet \vec{\mathbf{F}}$ for solenoidal $\vec{\mathbf{F}}$

   (c) $\operatorname{div}\big(\vec{\mathbf{F}} \times (\vec{\mathbf{F}} + \vec{\mathbf{G}})\big)$ for conservative $\vec{\mathbf{G}}$

7. A vector field $\vec{\mathbf{F}}$ is called a *curl field* if it can be expressed as $\vec{\mathbf{F}} = \mathit{curl}(\vec{\mathbf{G}})$ for some vector field $\vec{\mathbf{G}}$. In this case, $\vec{\mathbf{G}}$ is called a *vector potential* for $\vec{\mathbf{F}}$.

   (a) Explain why the following is true: if $\vec{\mathbf{G}}$ is a vector potential for a curl field $\vec{\mathbf{F}}$ and ϕ is a smooth scalar field, then $\vec{\mathbf{G}} + \nabla\phi$ is also a vector potential for $\vec{\mathbf{F}}$.

   Now, consider the vector field $\vec{\mathbf{F}} = \langle x^2e^{2y}, Aze^{2y}, (x-z)^2e^{2y}\rangle$ where A is a constant.

   (b) Only one choice for A makes $\vec{\mathbf{F}}$ a curl field. Find this value of A.

   (c) Using the value of A from part (b), find a vector potential for $\vec{\mathbf{F}}$ having special form $\vec{\mathbf{G}} = \langle G_1, 0, G_3\rangle$.

   (d) Repeat part (c), but find vector potentials with special forms $\langle 0, G_2, G_3\rangle$ and $\langle G_1, G_2, 0\rangle$. [Hint: Use the fact in part (a).]