Math 263 Assignment #4

Due in class on Wednesday 13 October 2004

1. Find and classify the critical points of each of the following functions:

   (a) $f(x, y, z) = x^2 + yz - x - 2y - z + 7$ (c) $f(x, y) = e^{-x^2-y^2}(1 - e^{x^2})$
   (b) $f(x, y) = (x + y)^3 - (x - y)(x - 5y)$ (d) $f(x, y) = 2 \sin x \cos y$

2. Find and classify all critical and singular points of $f(x, y) = 7\sqrt{x^2 + y^2} - 2(x-1)^2 + (x+1)^2$.

3. Find the (minimum) distance between the parabolas $\mathbf{r}_1(t) = \left\langle 0, 2t, -t^2 \right\rangle$, $-\infty < t < \infty$ and $\mathbf{r}_2(u) = \left\langle -u, 3, u^2 \right\rangle$, $-\infty < u < \infty$.

4. For what values of the constant k does the function $f(x, y) = kx^3 + x^2 + 2y^2 - 4x - 4y$ have: (a) no critical points; (b) exactly one critical point; (c) exactly two critical points? For parts (b) and (c), give the critical points (in terms of k).

5. Suppose the outside air temperature is given by

$$T(x, y, z) = -40 + \left(60 + 90z + \frac{5}{20 + x^2 + xy + y^2 - 2x + 4y}\right) e^{-z}$$

   for $z \geq 0$ (where $z = 0$ represents ground level). (a) Find any critical points. (b) Can a point at ground level have a global minimum or maximum temperature value? Why or why not? (c) Find the points of global minimum and maximum temperature value, or explain why such points do not exist.

6. [MATLAB] Consider the function $f(x, y, z) = \sin(xe^z) \cos y$.

   (a) Find algebraic expressions for the gradient $\nabla f(x, y, z)$ and the Hessian $\mathcal{H}(x, y, z)$.

   (b) Define the point $(a, b, c) = (\pi/e, \pi/2, 1)$, and show it is a critical point.

   (c) Using MATLAB, find the eigenvalues (and corresponding eigenvectors) of $\mathcal{H}(a, b, c)$, and verify (a, b, c) is a saddle point.

   (d) Find a vector (u, v, w) of length 0.1 parallel to one of the eigenvectors of $\mathcal{H}(a, b, c)$ such that $f(a + u, b + v, c + w) < f(a, b, c)$. [Check the inequality using MATLAB.]

   (e) Find a vector (u', v', w') of length 0.1 parallel to one of the eigenvectors of $\mathcal{H}(a, b, c)$ such that $f(a + u', b + v', c + w') > f(a, b, c)$. [Check the inequality using MATLAB.]

7. [MATLAB] For each of the following functions (i) find algebraic expressions for the gradient ∇f and the Hessian $\mathcal{H}$; (ii) write a MATLAB script that implements Newton's method for finding critical points; (iii) run your script with each of the given starting points and include in your answer the results of the first 10 iterations; and (iv) in those cases where the method appears to be converging, give a probable classification of the critical point.

   (a) $e^{-(x-1)^2-y^2} - e^{-(x+1)^2-y^2}$ starting at $(-1.5, 0.1)$, $(1.5, 0.1)$, and $(1.5, 0.2)$

   (b) $(y^2 + z^2 - 3)^2 + (x^2 + z^2 - 2)^2 + (x^2 - z)^2$ starting at $(1, 1.5, 1)$, $(0.5, 0.5, 0.5)$, and $(0.1, -0.1, 0.1)$

8. Recall that a function $f(x, y)$ is *harmonic* if it satisfies $f_{11}(x, y) + f_{22}(x, y) = 0$ for all x and y in its domain. Suppose f is a harmonic function with domain all of $\mathbb{R}^2$ and with $f_{11}(x, y) \neq 0$ for all x and y. Prove that f has no local minima or maxima.