Math 263 Final Exam Formulas (Fall, 2004)

PROJECTIONS

Scalar projection: $s = \dfrac{\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = |\vec{\mathbf{u}}| \cos\theta$

Vector projection: $\vec{\mathbf{u}}_{\vec{\mathbf{v}}} = \dfrac{\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|^2}\vec{\mathbf{v}} = \dfrac{\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}\hat{\mathbf{v}}$

OPTIMIZATION AND APPROXIMATION

Tangent Plane to surface $G(x,y,z) = 0$ at $P = (a,b,c)$: $\quad 0 = \langle x-a, y-b, z-c\rangle \bullet \nabla G(a,b,c)$

Linear Approximation $\quad f(\vec{\mathbf{x}}) \approx L(\vec{\mathbf{x}}) = f(\vec{\mathbf{a}}) + (\vec{\mathbf{x}} - \vec{\mathbf{a}}) \bullet \nabla f(\vec{\mathbf{a}})$

Quadratic Approximation

$$f(x,y) \approx Q(x,y) = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b) + \frac{f_{11}(a,b)}{2}(x-a)^2 + \frac{f_{22}(a,b)}{2}(y-b)^2 + f_{12}(a,b)(x-a)(y-b)$$

Hessian Test
(only for critical points)

$$H(x,y) = \begin{bmatrix} f_{11}(x,y) & f_{12}(x,y) \\ f_{12}(x,y) & f_{22}(x,y) \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}$$

- If $AD > B^2$ then
  - if $A < 0$ then local max;
  - if $A > 0$ then local min.
- If $AD < B^2$ then saddle point.

Newton's Method $\quad \vec{\mathbf{r}}_n = \vec{\mathbf{r}}_{n-1} - \big(H(\vec{\mathbf{r}}_{n-1})\big)^{-1} \nabla f(\vec{\mathbf{r}}_{n-1})$

CHAIN RULES

$$\frac{d}{dt} f\Big(x(t), y(t)\Big) = f_1\Big(x(t), y(t)\Big)x'(t) + f_2\Big(x(t), y(t)\Big)y'(t)$$

$$\frac{\partial}{\partial u} f\Big(x(u,v), y(u,v)\Big) = f_1\Big(x(u,v), y(u,v)\Big)\frac{\partial}{\partial u}x(u,v) + f_2\Big(x(u,v), y(u,v)\Big)\frac{\partial}{\partial u}y(u,v)$$

LINE INTEGRALS AND CONSERVATIVE FIELDS

Along parametrized curve $\mathcal{C}:\ \vec{\mathbf{r}} = \vec{\mathbf{r}}(t),\ a \le t \le b$,

$$\int_{\mathcal{C}} f(\vec{\mathbf{r}})\, ds = \int_a^b f(\vec{\mathbf{r}}(t)) \left|\frac{d\vec{\mathbf{r}}}{dt}\right| dt, \quad W = \int_{\mathcal{C}} dW = \int_{\mathcal{C}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}\big(\vec{\mathbf{r}}(t)\big) \bullet \frac{d\vec{\mathbf{r}}}{dt} dt$$

$\vec{\mathbf{F}}$ conservative on set U means $\vec{\mathbf{F}} = \mathbf{grad}(\phi)$ on U.

If $\vec{\mathbf{F}} = \langle F_1, F_2, F_3\rangle$ is conservative on U, then $\mathbf{curl}(\vec{\mathbf{F}}) = \vec{\mathbf{0}}$ on U.

SURFACE NORMALS AND AREA ELEMENTS

Parametric surface $\vec{\mathbf{r}} = \vec{\mathbf{r}}(u,v)$: $\quad \vec{\mathbf{n}} = \pm\left(\dfrac{\partial \vec{\mathbf{r}}}{\partial u} \times \dfrac{\partial \vec{\mathbf{r}}}{\partial v}\right) \quad d\vec{\mathbf{S}} = \pm\left(\dfrac{\partial \vec{\mathbf{r}}}{\partial u} \times \dfrac{\partial \vec{\mathbf{r}}}{\partial v}\right) du\, dv \quad dS = \left|\dfrac{\partial \vec{\mathbf{r}}}{\partial u} \times \dfrac{\partial \vec{\mathbf{r}}}{\partial v}\right| du\, dv$

Surface with equation $z = g(x,y)$: $\quad \vec{\mathbf{n}} = \pm\Big(-g_1\,\vec{\mathbf{\imath}} - g_2\,\vec{\mathbf{\jmath}} + \vec{\mathbf{k}}\Big) \quad d\vec{\mathbf{S}} = \pm\Big(-g_1\,\vec{\mathbf{\imath}} - g_2\,\vec{\mathbf{\jmath}} + \vec{\mathbf{k}}\Big)\, dx\, dy \quad dS = \sqrt{(g_1)^2 + (g_2)^2 + 1}\, dx\, dy$

Level surface $G(x,y,z) = 0$: $\quad \vec{\mathbf{n}} = \pm\nabla G(x,y,z) \quad d\vec{\mathbf{S}} = \pm\dfrac{\nabla G(x,y,z)}{|\partial G/\partial z|} dx\, dy \quad dS = \left|\dfrac{\nabla G(x,y,z)}{\partial G/\partial z}\right| dx\, dy$

Other projections: $\quad d\vec{\mathbf{S}} = \dfrac{\vec{\mathbf{n}}}{|\vec{\mathbf{n}} \bullet \vec{\mathbf{k}}|}\, dx\, dy = \dfrac{\vec{\mathbf{n}}}{|\vec{\mathbf{n}} \bullet \vec{\mathbf{\jmath}}|}\, dx\, dz = \dfrac{\vec{\mathbf{n}}}{|\vec{\mathbf{n}} \bullet \vec{\mathbf{\imath}}|}\, dy\, dz \quad dS = |d\vec{\mathbf{S}}|$

POLAR COORDINATES

$x = r\cos\theta, \quad y = r\sin\theta \qquad r = \sqrt{x^2+y^2}, \quad \tan\theta = y/x \qquad dA = r\, dr\, d\theta$

CYLINDRICAL COORDINATES

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \qquad r = \sqrt{x^2+y^2}, \quad \tan\theta = y/x, \quad z = z$

$dV = r\, dr\, d\theta\, dz \qquad$ Surface area element (on $r = a$): $\quad dS = a\, d\theta\, dz$

SPHERICAL COORDINATES

$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi \qquad \rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}, \quad \tan\phi = \dfrac{r}{z} = \dfrac{\sqrt{x^2+y^2}}{z}, \quad \tan\theta = \dfrac{y}{x}$

$dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta \qquad$ Surface area element (on $\rho = a$): $\quad dS = a^2\sin\phi\, d\phi\, d\theta$

Adapted from R. A. Adams, *Calculus, Several Variables,* Addison-Wesley, 2003.

GRADIENT, DIVERGENCE, AND CURL

$\nabla = \vec{\mathbf{i}}\frac{\partial}{\partial x} + \vec{\mathbf{j}}\frac{\partial}{\partial y} + \vec{\mathbf{k}}\frac{\partial}{\partial z}$ ("del" or "nabla" ooperator)

$\vec{\mathbf{F}}(x,y,z) = F_1(x,y,z)\,\vec{\mathbf{i}} + F_2(x,y,z)\,\vec{\mathbf{j}} + F_3(x,y,z)\,\vec{\mathbf{k}}$

$\nabla\phi(x,y,z) = \mathbf{grad}\,\phi(x,y,z) = \frac{\partial\phi}{\partial x}\vec{\mathbf{i}} + \frac{\partial\phi}{\partial y}\vec{\mathbf{j}} + \frac{\partial\phi}{\partial z}\vec{\mathbf{k}}$

$\nabla \bullet \vec{\mathbf{F}}(x,y,z) = \mathbf{div}\ \vec{\mathbf{F}}(x,y,z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$\nabla \times \vec{\mathbf{F}}(x,y,z) = \mathbf{curl}\,\vec{\mathbf{F}}(x,y,z) = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{\mathbf{k}}$$

$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \bullet (\vec{\mathbf{F}} \times \vec{\mathbf{G}}) = (\nabla \times \vec{\mathbf{F}}) \bullet \vec{\mathbf{G}} - \vec{\mathbf{F}} \bullet (\nabla \times \vec{\mathbf{G}})$

$\nabla \bullet (\phi\vec{\mathbf{F}}) = (\nabla\phi) \bullet \vec{\mathbf{F}} + \phi\,(\nabla \bullet \vec{\mathbf{F}})$

$\nabla \times (\vec{\mathbf{F}} \times \vec{\mathbf{G}}) = \vec{\mathbf{F}}(\nabla \bullet \vec{\mathbf{G}}) - \vec{\mathbf{G}}\,(\nabla \bullet \vec{\mathbf{F}}) - (\vec{\mathbf{F}} \bullet \nabla)\vec{\mathbf{G}} + (\vec{\mathbf{G}} \bullet \nabla)\vec{\mathbf{F}}$

$\nabla \times (\phi\vec{\mathbf{F}}) = (\nabla\phi) \times \vec{\mathbf{F}} + \phi\,(\nabla \times \vec{\mathbf{F}})$

$\nabla(\vec{\mathbf{F}} \bullet \vec{\mathbf{G}}) = \vec{\mathbf{F}} \times (\nabla \times \vec{\mathbf{G}}) + \vec{\mathbf{G}} \times (\nabla \times \vec{\mathbf{F}}) + (\vec{\mathbf{F}} \bullet \nabla)\vec{\mathbf{G}} + (\vec{\mathbf{G}} \bullet \nabla)\vec{\mathbf{F}}$

$\nabla \times (\nabla\phi) = \vec{\mathbf{0}}$ ($\mathbf{curl\ grad} = \vec{\mathbf{0}}$)

$\nabla \bullet (\nabla \times \vec{\mathbf{F}}) = 0$ ($\mathbf{div\ curl} = 0$)

VERSIONS OF THE FUNDAMENTAL THEOREM OF CALCULUS

$\int_a^b f'(t)\,dt = f(b) - f(a)$ (the one-dimensional **Fundamental Theorem**)

$\int_C \nabla\phi \bullet d\vec{\mathbf{r}} = \phi(\vec{\mathbf{r}}(b)) - \phi(\vec{\mathbf{r}}(a))$, if C is the curve $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$, $a \le t \le b$

$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$, where C is the positively oriented boundary of R **(Green's Theorem)**

$\iint_S \mathbf{curl}\,\vec{\mathbf{F}} \bullet \widehat{\mathbf{N}}\,dS = \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$, where C is the oriented boundary of S **(Stokes's Theorem)**

$\iiint_D \mathbf{div}\,\vec{\mathbf{F}}\,dV = \oiint_S \vec{\mathbf{F}} \bullet \widehat{\mathbf{N}}\,dS$, where S is the bounding surface of D, with outward unit normal $\widehat{\mathbf{N}}$ **(Divergence Theorem)**

TRIGONOMETRIC IDENTITIES

$\sin^2 x + \cos^2 x = 1$

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\sec^2 x = 1 + \tan^2 x$

$\csc^2 x = 1 + \cot^2 x$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$\cos^2 x = \frac{1 + \cos 2x}{2}$

INTEGRALS

$\int \sec^2 x\,dx = \tan x + C$

$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$

$\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

$\int \tan x\,dx = \ln|\sec x| + C$

$\int \sin^3 x\,dx = \frac{1}{3}\cos^3 x - \cos x + C$

$\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C$

$\int \tan^2 x\,dx = \tan x - x + C$

$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \quad (a > 0)$

$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{x + a}{x - a}\right| + C \quad (a > 0)$

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C \quad (a > 0,\ |x| < a)$

$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

$\int \sqrt{x^2 \pm a^2}\,dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

$\int_0^{\pi/2} \sin x\,dx = \int_0^{\pi/2} \cos x\,dx = 1$

$\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$

$\int_0^{\pi/2} \sin^3 x\,dx = \int_0^{\pi/2} \cos^3 x\,dx = \frac{2}{3}$

$\int_0^{\pi/2} \sin^4 x\,dx = \int_0^{\pi/2} \cos^4 x\,dx = \frac{3\pi}{16}$

$\int_0^{\pi/2} \sin^5 x\,dx = \int_0^{\pi/2} \cos^5 x\,dx = \frac{8}{15}$

$\int_0^{\pi/2} \sin^6 x\,dx = \int_0^{\pi/2} \cos^6 x\,dx = \frac{5\pi}{32}$

Adapted from R. A. Adams, *Calculus, Several Variables,* Addison-Wesley, 2003.