[25] **1.** Find and classify all critical points of

$$f(x, y) = x^3 - 3xy^2 - 3x^2 - 3y^2.$$

Calculation gives

$$\begin{aligned} f(x,y) &= x^3 - 3xy^2 - 3x^2 - 3y^2 & f_x(x,y) &= 3x^2 - 3y^2 - 6x & f_{xx}(x,y) &= 6x - 6 \\ & & f_y(x,y) &= -6xy - 6y & f_{yy}(x,y) &= -6x - 6 \\ & & & & f_{xy}(x,y) &= -6y \end{aligned}$$

At a critical point both $f_x(x, y) = 0$ and $f_y(x, y) = 0$, i.e.,

$$(1) \quad 3(x^2 - y^2 - 2x) = 0, \qquad (2) \quad -6y(x + 1) = 0.$$

From equation (2), we get two cases: $x = -1$ or $y = 0$.

Case $x = -1$**.** Here (1) reduces to $y^2 = 3$, so $y = \pm\sqrt{3}$. This gives two CP's:

$$(-1, -\sqrt{3}), \qquad (-1, \sqrt{3}).$$

Case $y = 0$**.** Here (1) reduces to $0 = x^2 - 2x = x(x - 2)$, so $x = 0$ or $x = 2$. This gives two CP's:

$$(0, 0), \qquad (2, 0).$$

Here is a table giving the classification of each of the four critical points.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0, 0)$	$(-6) \times (-6) - 0^2 > 0$	-6	local max
$(2, 0)$	$6 \times (-18) - 0^2 < 0$		saddle point
$(-1, \sqrt{3})$	$(-12) \times 0 - (-6\sqrt{3})^2 < 0$		saddle point
$(-1, -\sqrt{3})$	$(-12) \times 0 - (6\sqrt{3})^2 < 0$		saddle point

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[25] **2.** Consider the equation

$$(*) \qquad (x-1)^2 - 2y^2(2-y^2) + (z-1)^2 - z^3 + 1 = 0.$$

(a) Assuming that $(*)$ defines z as a function of x and y, find the gradient $\nabla z = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle$.

(b) If $x = 1 + \sqrt{3}\cos t$ and $y = \sqrt{3}\sin t$, use the result in (a) to calculate $\frac{d}{dt} z(x(t), y(t))$ at the point where $(x, y, z) = (1+\sqrt{2}, 1, 1)$.

(a) By implicit differentiation with respect to x, we get:

$$2(x-1) + 2(z-1)\frac{\partial z}{\partial x} - 3z^2 \frac{\partial z}{\partial x} = 0$$

and solving for $\partial z/\partial x$ gives

$$\frac{\partial z}{\partial x} = \frac{2(x-1)}{3z^2 - 2(z-1)}$$

Similarly, implicit differentiation with respect to y gives:

$$-4y(2-y^2) + 4y^3 + 2(z-1)\frac{\partial z}{\partial y} - 3z^2 \frac{\partial z}{\partial y} = 0$$

and solving for $\partial z/\partial y$ gives

$$\frac{\partial z}{\partial y} = \frac{8y^3 - 8y}{3z^2 - 2(z-1)}$$

This gives the gradient

$$\nabla z(x, y) = \left\langle \frac{2(x-1)}{3z^2 - 2(z-1)}, \frac{8y^3 - 8y}{3z^2 - 2(z-1)} \right\rangle$$

(b) Since $dx/dt = -\sqrt{3}\sin t$ and $dy/dt = \sqrt{3}\cos t$, by the chain rule, we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = -\frac{2(x-1)}{3z^2 - 2(z-1)}\sqrt{3}\sin t + \frac{8y^3 - 8y}{3z^2 - 2(z-1)}\sqrt{3}\cos t$$

Since the point has $x = 1+\sqrt{2}$ and $y = 1$, we clearly have $\sqrt{3}\cos t = \sqrt{2}$ and $\sqrt{3}\sin t = 1$. Substituting these values and the values for x, y, z into the above equation, we get

$$\frac{dz}{dt} = -\frac{2\sqrt{2}}{3}(1) + \frac{0}{3}\sqrt{2} = -\frac{2\sqrt{2}}{3}$$

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[25] **3.** Consider the set D in the xy-plane defined by

$$D: \qquad x \geq 0,\ y \geq 0,\ x+y \leq 1.$$

Find the maximum value of f on D, and the point(s) where it occurs, given

$$f(x,y) = x^2 y^4 e^{-x-y}.$$

Notice that $f(x,y) = \left(x^2 e^{-x}\right)\left(y^4 e^{-y}\right)$. Use the reduction-of-dimension strategy.

(2D) Interior Points: In the set where $x > 0$, $y > 0$, $x + y < 1$, calculation gives

$$\text{(1)} \qquad \frac{\partial f}{\partial x} = y^4 e^{-y}\left[2xe^{-x} - x^2 e^{-x}\right] = x(2-x)y^4 e^{-x-y},$$

$$\text{(2)} \qquad \frac{\partial f}{\partial y} = x^2 e^{-x}\left[4y^3 e^{-y} - y^4 e^{-y}\right] = x^2 y^3 (4-y) e^{-x-y}.$$

To get $\partial f/\partial x = 0$ requires either $x = 0$ or $x = 2$ or $y = 0$, but no points in the interior of D satisfy any of these three conditions. So this case produces no points of interest.

(1D) Left Edge: At all points where $x = 0$ and $0 < y < 1$, we have $f(0,y) = 0$.

(1D) Bottom Edge: At all points where $y = 0$ and $0 < x < 1$, we have $f(x,0) = 0$.

(1D) Top Edge: Here $0 < x < 1$ and $y = 1 - x$, and $f(x, 1-x) = x^2(1-x)^4 e^{-1} \overset{\text{def}}{=} g(x)$. Calculation (product rule) gives
$eg'(x) = [2x](1-x)^4 + x^2[4(1-x)^3(-1)] = 2x(1-x)^3\left[(1-x) - 2x\right] = 2x(1-x)^3(1-3x)$.
The only CP for g obeying $0 < x < 1$ is $x = 1/3$, which corresponds to $(1/3, 2/3)$ on the top edge of D. At this point,

$$f(1/3, 2/3) = \frac{2^4}{3^6} e^{-1} = \frac{16}{729e}.$$

(0D) Corner Points: Set D is a triangle, with corners at $(0,0)$, $(1,0)$, $(0,1)$. At each corner point f has the value 0.

Summary: Among all points of interest identified above, the one with the largest function value lies on the top edge of D:

$$\text{Maximum value: } \frac{16}{729e} = f(1/3, 2/3).$$

Critical Points: To find **all** CP's for f in $\mathbb{R}^2$, use equation (1) to eliminate one variable and study the reduced form of (2). Equation (1) may hold for three reasons: either $x = 0$, or $y = 0$, or $x = 2$.

1. If $x = 0$, then (2) holds for all real y. So all points of the form $(0, y)$, $y \in \mathbb{R}$, are CP's.
2. If $y = 0$, then (2) holds for all real x. So all points of the form $(x, 0)$, $x \in \mathbb{R}$, are CP's.
3. If $x = 2$, then (2) holds when either $y = 0$ or $y = 4$. The point $(2, 0)$ has already been catalogued in case 2, but the CP at $(2, 4)$ is new.

Thus f has infinitely many CP's: the two lines $x = 0$ and $y = 0$ and the isolated point $(2, 4)$.

Discussion [not required for credit]: At the maximizing point $(1/3, 2/3)$,

$$\nabla f(1/3, 2/3) = \cdots = \frac{80}{729e}\left\langle 1,\, 1\right\rangle.$$

This is not zero (boundary extrema need not be CP's), but it does point in the outward normal direction to the boundary of D at the point of interest.

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[25] **4.** Let T be the triangle in the xy-plane bounded by the lines

$$x = 0, \quad y = 1, \quad y = x.$$

(a) Let $I = \iint_T f(x,y)\,dA$. Express I as an iterated integral in two different ways: one where the inner integral involves dx, and one where the inner integral involves dy.
(Express your answers in terms of the general function f.)

(b) Given $f(x,y) = e^y/y$, find the average value of f on T.

Recall: The average value of a function f on a plane region T is, by definition,

$$\overline{f} = \frac{1}{\text{Area}(T)} \iint_T f(x,y)\,dA.$$

(a) Projecting T along the x-direction onto the y-axis fills the interval $0 \le y \le 1$; the horizontal filament at level y runs from $x = 0$ to $x = y$. Thus

$$\iint_T f(x,y)\,dA = \int_0^1 \int_0^y f(x,y)\,dx\,dy.$$

Projecting T along the y-direction onto the x-axis fills the interval $0 \le x \le 1$; the vertical filament at position x runs from $y = x$ to $y = 1$. Thus

$$\iint_T f(x,y)\,dA = \int_0^1 \int_x^1 f(x,y)\,dy\,dx.$$

(b) The region T is a right triangle with both base and height of length 1, so Area$(T) = 1/2$.

When $f(x,y) = e^y/y$, it is convenient to have an inner integral in terms of x. Thus

$$\iint_T \frac{e^y}{y}\,dA = \int_0^1 \int_0^y \frac{e^y}{y}\,dx\,dy = \int_0^1 \frac{e^y}{y}\Big[x\Big]_{x=0}^{y}\,dy$$

$$= \int_0^1 e^y\,dy = \Big[e^y\Big]_{y=0}^{1} = e - 1.$$

Hence $\overline{f} = (e-1)/\text{Area}(T) = 2e - 2.$

The End

This examination has 5 pages including this cover

The University of British Columbia

Midterm Examination – 28 Oct 2004

Mathematics 263

Multivariable and Vector Calculus

Closed book examination | Time: 50 minutes

Name______________________________ **Signature** ______________________________

Student Number____________________

Special Instructions:

To receive full credit, all answers must be supported with clear and correct derivations. No calculators, notes, or other aids are allowed. A formula sheet is provided with the test.

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Total		100