Quadratic Approximation for $f(x, y)$

For a function $F(t)$ of a single variable, the quadratic approximation at a is given by

$$F(t) \approx F(a) + (t-a)F'(a) + (t-a)^2 \frac{F''(a)}{2}$$

In particular, at $a = 0$, the approximation for $t = 1$ is

$$F(1) \approx F(0) + F'(0) + \frac{F''(0)}{2} \tag{1}$$

To find a quadratic approximation of a two-variable function $f(x, y)$, we can use a clever trick to reduce it to the one-variable case. Let us fix a, b, x, and y and—treating these as constant values—define the one-variable function

$$F(t) = f\big(a + t(x-a), b + t(y-b)\big)$$

Because this function satisfies $F(1) = f(x, y)$, we can approximate $f(x, y)$ by approximating $F(1)$ using formula (1).

To use this formula, we'll need to calculate $F(0)$, $F'(0)$, and $F''(0)$. Of course, the definition of $F(t)$ implies $F(0) = f(a, b)$, but to calculate the others, we'll need to find $F'(t)$ and $F''(t)$ using the (multivariable) chain rule.

First, let's find $F'(t)$. Using either the first version of the chain rule or the general method from class, we have

$$\begin{aligned} F'(t) &= \frac{d}{dt} f\big(a + t(x-a), b + t(y-b)\big) \\ &= f_1\big(a + t(x-a), b + t(y-b)\big) \frac{d}{dt}\big(a + t(x-a)\big) + f_2\big(a + t(x-a), b + t(y-b)\big) \frac{d}{dt}\big(b + t(y-b)\big) \\ &= f_1\big(a + t(x-a), b + t(y-b)\big)(x-a) + f_2\big(a + t(x-a), b + t(y-b)\big)(y-b) \end{aligned}$$

Therefore, substituting in $t = 0$, we have

$$F'(0) = f_1(a, b)(x-a) + f_2(a, b)(y-b)$$

Second, let's find $F''(t)$ by differentiating the formula for $F'(t)$ calculated above

$$\begin{aligned} F''(t) &= \frac{d}{dt} F'(t) \\ &= \frac{d}{dt}\Big(f_1\big(a + t(x-a), b + t(y-b)\big)(x-a) + f_2\big(a + t(x-a), b + t(y-b)\big)(y-b) \Big) \\ &= \frac{d}{dt}\Big(f_1\big(a + t(x-a), b + t(y-b)\big) \Big)(x-a) \\ &\qquad + \frac{d}{dt}\Big(f_2\big(a + t(x-a), b + t(y-b)\big) \Big)(y-b) \end{aligned} \tag{2}$$

To get further, we'll need to apply the first version of the chain rule or the general method from class to calculate:

$$\begin{aligned}\frac{d}{dt}\Big(f_1\big(a+t(x-a),b+t(y-b)\big)\Big) &= f_{11}\big(a+t(x-a),b+t(y-b)\big)\frac{d}{dt}\big(a+t(x-a)\big)\\ &\quad + f_{12}\big(a+t(x-a),b+t(y-b)\big)\frac{d}{dt}\big(b+t(y-b)\big)\\ &= f_{11}\big(a+t(x-a),b+t(y-b)\big)(x-a)\\ &\quad + f_{12}\big(a+t(x-a),b+t(y-b)\big)(y-b)\end{aligned} \tag{3}$$

Similarly, we can calculate

$$\begin{aligned}\frac{d}{dt}\Big(f_2\big(a+t(x-a),b+t(y-b)\Big) &= f_{21}\big(a+t(x-a),b+t(y-b)\big)(x-a)\\ &\quad + f_{22}\big(a+t(x-a),b+t(y-b)\big)(y-b)\end{aligned} \tag{4}$$

Subsituting (3) and (4) into (2), we get

$$\begin{aligned}F''(t) = &\Big(f_{11}\big(a+t(x-a),b+t(y-b)\big)(x-a) + f_{12}\big(a+t(x-a),b+t(y-b)\big)(y-b)\Big)(x-a)\\ &+\Big(f_{21}\big(a+t(x-a),b+t(y-b)\big)(x-a) + f_{22}\big(a+t(x-a),b+t(y-b)\big)(y-b)\Big)(y-b)\end{aligned}$$

and so, for $t=0$, this simplifies to

$$F''(0) = f_{11}(a,b)(x-a)^2 + f_{12}(a,b)(y-b)(x-a) + f_{21}(a,b)(x-a)(y-b) + f_{22}(a,b)(y-b)^2$$

Substituting all of these back into formula (1), we get

$$\begin{aligned}f(x,y) = F(1) \approx\ & f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)\\ &+ \tfrac{1}{2}f_{11}(a,b)(x-a)^2 + \tfrac{1}{2}f_{12}(a,b)(y-b)(x-a)\\ &+ \tfrac{1}{2}f_{21}(a,b)(x-a)(y-b) + \tfrac{1}{2}f_{22}(a,b)(y-b)^2\end{aligned}$$

If f_{12} and f_{21} exist and are continuous at (a,b), then by a theorem from class they are equal, so we can "simplify" this expression to its final form:

$$\begin{aligned}f(x,y) \approx\ & f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)\\ &+ \tfrac{1}{2}f_{11}(a,b)(x-a)^2 + f_{12}(a,b)(x-a)(y-b) + \tfrac{1}{2}f_{22}(a,b)(y-b)^2\end{aligned}$$