15.5 Question #17 (p. 936)

Find the total charge on the surface

$$\mathbf{r} = e^u \cos v \,\vec{\imath} + e^u \sin v \,\vec{\jmath} + u \,\vec{\mathbf{k}}, \qquad (0 \le u \le 1,\ 0 \le v \le \pi).$$

if the charge density on the surface is $\delta = \sqrt{1 + e^{2u}}$.

We want to find $\iint_{\mathcal{S}} \delta \, dS$. We already have a parameterization for the surface $\mathcal{S}$, so we only need to calculate the area element dS. Since

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} &= \langle e^u \cos v, e^u \sin v, 1 \rangle \times \langle -e^u \sin v, e^u \cos v, 0 \rangle \\
&= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ e^u \cos v & e^u \sin v & 1 \\ -e^u \sin v & e^u \cos v & 0 \end{vmatrix} = \left\langle -e^u \cos v, -e^u \sin v, e^{2u} \right\rangle
\end{aligned}$$

we have

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du \, dv = \sqrt{e^{2u} \cos^2 v + e^{2u} \sin^2 v + e^{4u}} = \sqrt{e^{2u} + e^{4u}} = e^u \sqrt{1 + e^{2u}}$$

Therefore,

$$\begin{aligned}
\iint_{\mathcal{S}} \delta \, dS &= \iint_{\mathcal{S}} \sqrt{1 + e^{2u}} e^u \sqrt{1 + e^{2u}} \, dS = \iint_{\mathcal{S}} (1 + e^{2u}) e^u \, dS \\
&= \int_0^1 du \int_0^\pi dv \big[e^u + e^{3u} \big] = \int_0^1 du \big[(e^u + e^{3u}) \pi \big] = \pi \left(e^u + \frac{e^{3u}}{3} \right) \Bigg|_{u=0}^{u=1} \\
&= \pi \left(e + \frac{e^3}{3} - 1 - \frac{1}{3} \right) = \pi \left(e + \frac{e^3}{3} - \frac{4}{3} \right)
\end{aligned}$$