Math 263 Sample Final Exam: Solution #8

8. (a) By the Divergence Theorem, the total flux of $\vec{\mathbf{F}}$ outward through $\mathcal{S}$ can be rewritten as a volume integral:

$$\iint_{\mathcal{S}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = \iiint_{\mathcal{R}} \nabla \bullet \vec{\mathbf{F}}\, dV$$

where $\nabla \bullet \vec{\mathbf{F}} = 2x + 2 - 2 = 2x$.
However, the triple integral

$$\iiint_{\mathcal{R}} 2x\, dV$$

is the integral of an odd function of f over a region $\mathcal{R}$ that is symmetric across the yz-plane (that is, the plane $x = 0$). By symmetry, it has value 0.
Alternatively, if (like me) you didn't notice that simplification and instead iterated the integral in Cartesian coordinates, you would get something like:

$$\iiint_{\mathcal{R}} 2x\, dV = \int_{-1}^{1} dx \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} dy \int_{0}^{y} dz\, [2x] = \int_{-1}^{1} dx \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} dy\, [2xy]$$

$$= \int_{-1}^{1} dx\, [xy^2]\Big|_{y=1-\sqrt{1-x^2}}^{y=1+\sqrt{1-x^2}} = \int_{-1}^{1} 4x\sqrt{1-x^2}\, dx = -2\int_{u=0}^{u=0} \sqrt{u}\, du = 0$$

where we used the substitution $u = 1 - x^2$ at the end.
In any event, this gives the final answer of 0.

(b) There are several valid approaches. One way is to use the fact that the total flux out of $\mathcal{S}$ is 0, so we must have

$$0 = \iint_{\mathcal{S}_{\text{top}}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} + \iint_{\mathcal{S}_{\text{side}}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} + \iint_{\mathcal{S}_{\text{bottom}}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}}$$

where all surfaces are oriented with the outward normal. If the flux out of the top and out of the bottom are easy to calculate, we can easily calculate the desired integral out of the (vertical) side.
The surface $\mathcal{S}_{\text{top}}$ has $z = y$, so we can parametrize it as:

$$\vec{\mathbf{r}}(x, y) = \langle x, y, y \rangle, \qquad -1 \le x \le 1,\ 1 - \sqrt{1-x^2} \le y \le 1 + \sqrt{1-x^2}$$

Using the formula for $d\vec{\mathbf{S}}$ for a surface given by $z = g(x, y)$ (where $g(x, y) = y$ here), we have

$$d\vec{\mathbf{S}} = \pm \langle 0, -1, 1 \rangle\, dx\, dy$$

On the top, we want the $d\vec{\mathbf{S}}$ that points up, so we'll take $d\vec{\mathbf{S}} = + \langle 0, -1, 1 \rangle\, dx\, dy$. Thus,

$$\iint_{\mathcal{S}_{\text{top}}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = \int_{-1}^{1} dx \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} dy\, \left\langle x^2, 2y, -2y \right\rangle \bullet \langle 0, -1, 1 \rangle$$

$$= \int_{-1}^{1} dx \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} dy\, [-4y] = \int_{-1}^{1} dx\, [-2y^2]\Big|_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}}$$

$$= -2\int_{-1}^{1} 4\sqrt{1-x^2} dx = -8\left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right)\Bigg|_{x=-1}^{x=1} = -4\pi$$

On the other hand, the surface $\mathcal{S}_{\text{bottom}}$ has $z = 0$ and outward (i.e., downward) unit normal $\langle 0, 0, -1\rangle$, so we have

$$\iint\limits_{\mathcal{S}_{\text{bottom}}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = \iint_D \left\langle x^2, 2y, 0\right\rangle \bullet \langle 0, 0, -1\rangle \, dx\, dy = 0$$

Therefore, the final answer is

$$\iint\limits_{\mathcal{S}_{\text{side}}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = -(-4\pi) - 0 = 4\pi$$