Math 221 (101) Matrix Algebra November 22, 2002

Quiz #9

DUE: Friday, November 29 at 9:00am

Rules:

1. Quizzes are due at the beginning of class on the due date. Late quizzes will not receive credit. Remember to include your name and student number on your answers.
2. You may work with other Math 221 students and use materials such as textbooks, notes, calculators, or computers. However, you must write up your own original, complete solutions. Copying all or part of an answer is not allowed.
3. Show all work. Explain your solutions clearly. You may write in pencil if you'd like.
4. The maximum possible score is 20.

Solve the following five problems, showing all work. Don't forget to put your name and student number on your answers.

1. Let T be a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$$

and let $\mathfrak{B}=\left\{\begin{bmatrix}3\\1\end{bmatrix},\begin{bmatrix}-2\\6\end{bmatrix}\right\}$.

(a) Calculate the standard matrix for T.

(b) Show that $\mathfrak{B}$ is an orthogonal basis for $\mathbb{R}^2$.

(c) Calculate $[T]_{\mathfrak{B}}$. [Hint: The first column is $[T(\mathbf{b}_1)]_{\mathfrak{B}}$. Calculate $T(\mathbf{b}_1)$ using part (a). Then, calculate $[T(\mathbf{b}_1)]_{\mathfrak{B}}$ using Theorem 6.5. Do the same for the second column.]

2. Let A be given by

$$A=\begin{bmatrix}1&-1&-1&0\\4&8&5&9\\2&2&1&3\end{bmatrix}$$

(a) What does $\operatorname{Col} A$ "look like" as a subspace of $\mathbb{R}^3$? That is, is it a point, a line, a plane, or all of $\mathbb{R}^3$?

(b) Find a basis for the orthogonal complement of $\operatorname{Col} A$.

3. Consider the matrix

$$A=\begin{bmatrix}1&0&2\\4&3&6\\1&0&-2\\2&5&10\end{bmatrix}$$

The rank of A is known to be 3.

(a) Find a least-squares solution for

$$A\mathbf{x} = \begin{bmatrix} 20 \\ 3 \\ 4 \\ -7 \end{bmatrix}$$

(b) Use the Gram-Schmidt process to calculate an orthogonal basis for $\operatorname{Col} A$.

(c) Give an orthonormal basis for $\operatorname{Col} A$.

(d) Give a QR factorization of A or explain why none exists.

4. Consider the symmetric matrix

$$A = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(a) Find an orthogonal diagonalization of A. That is, give a diagonal matrix D and an orthogonal matrix P (one with ortho*normal* columns) such that $A = PDP^{-1}$.

(b) Give the spectral decomposition of A explicitly in the form of a linear combination of three rank 1 matrices.

(c) Is A positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite?

5. In Quiz #4, problem 4(b), we modeled the evolution of a bacterial colony by a linear difference equation. Recall that, at 40°C, the number of wild type (w_k) and heat sensitive (h_k) bacteria (in thousands) after k hours was given by the difference equation

$$\mathbf{x}_{k+1} = \begin{bmatrix} w_{k+1} \\ h_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} w_k \\ h_k \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x}_k$$

We will use eigenvalues and eigenvectors to describe the evolution of this population more completely.

(a) Diagonalize A. That is, find an invertible P and a diagonal D such that $A = PDP^{-1}$.

(b) Give an eigenvector $\mathbf{u}$ for eigenvalue 1 and an eigenvector $\mathbf{v}$ for eigenvalue 4. If the initial colony is given by $\mathbf{x}_0 = c\mathbf{u}$ for some scalar c, how does the colony evolve over time? What if the initial colony is given by $\mathbf{x}_0 = c\mathbf{v}$?

(c) Explain why $\{\mathbf{u}, \mathbf{v}\}$ is a basis for $\mathbb{R}^2$.

(d) If an initial population vector $\mathbf{x}_0$ can be expressed as a linear combination $\mathbf{x}_0 = c\mathbf{u} + d\mathbf{v}$ for scalars c and d, explain why $\mathbf{x}_1 = c\mathbf{u} + 4d\mathbf{v}$. Explain why $\mathbf{x}_2 = c\mathbf{u} + 4^2 d\mathbf{v}$. Give a similar formula for $\mathbf{x}_k$ for general k.

(e) Express the initial population vector $\mathbf{x}_0 = (1, 0)$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Use the result from part (d) to give an explicit expression for the vector $\mathbf{x}_k$ (where the entries depend on k). Check that your formula gives the correct answer $\mathbf{x}_3 = (640, 210)$.