Quiz #9 Solutions

1. (a) We already have $T(\mathbf{e}_1) = (1, 2)$. To find $T(\mathbf{e}_2)$, we note that

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

so

$$T(\mathbf{e}_2) = \frac{1}{2}\left(T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right) = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

so the standard matrix A for T is

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

(b) Since $(3, 1) \cdot (-2, 6) = 3(-2) + 1(6) = 0$, the vectors are orthogonal. Since they are nonzero, they are linearly independent, and two linearly independent vectors in $\mathbb{R}^2$ form a basis for $\mathbb{R}^2$.

(c) We need to calculate

$$[T]_{\mathfrak{B}} = \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathfrak{B}} & [T(\mathbf{b}_2)]_{\mathfrak{B}} \end{bmatrix}$$

For the first column,

$$T(\mathbf{b}_1) = A\mathbf{b}_1 = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Using Theorem 6.5, we can write $(3, 5)$ as the linear combination:

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{\begin{bmatrix} 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 6 \end{bmatrix}}{\begin{bmatrix} -2 \\ 6 \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix}} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \frac{14}{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{24}{40}\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \frac{7}{5}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

giving a coordinate vector

$$[T(\mathbf{b}_1)]_{\mathfrak{B}} = \begin{bmatrix} 7/5 \\ 3/5 \end{bmatrix}$$

Similarly, for the second column,

$$T(\mathbf{b}_2) = A\mathbf{b}_2 = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -10 \end{bmatrix}$$

Using Theorem 6.5, we can write $(-2, -10)$ as the linear combination:

$$\begin{bmatrix} -2 \\ -10 \end{bmatrix} = \frac{\begin{bmatrix} -2 \\ -10 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} -2 \\ -10 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 6 \end{bmatrix}}{\begin{bmatrix} -2 \\ 6 \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix}} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \frac{-16}{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{-56}{40}\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \frac{-8}{5}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{-7}{5}\begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

giving a coordinate vector

$$[T(\mathbf{b}_1)]_{\mathfrak{B}} = \begin{bmatrix} -8/5 \\ -7/5 \end{bmatrix}$$

That gives a final answer of

$$[T]_{\mathfrak{B}} = \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathfrak{B}} & [T(\mathbf{b}_2)]_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} 7/5 & -8/5 \\ 3/5 & -7/5 \end{bmatrix}$$

2. (a) We need to find the dimension of $\operatorname{Col} A$. Row reducing the matrix to echelon form, we find

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 4 & 8 & 5 & 9 \\ 2 & 2 & 1 & 3 \end{bmatrix} \xrightarrow{R2 \to R2 - 4R1} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 12 & 9 & 9 \\ 2 & 2 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - 2R1} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 12 & 9 & 9 \\ 0 & 4 & 3 & 3 \end{bmatrix} \xrightarrow{R3 \to R3 - \frac{1}{3}R2} \begin{bmatrix} \boxed{1} & -1 & -1 & 0 \\ 0 & \boxed{12} & 9 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are two pivot columns, so the basis consists of two vectors. Therefore, $\operatorname{Col} A$ is a subspace of dimension 2, a plane through the origin.

(b) By Theorem 6.3, $(\operatorname{Col} A)^{\perp} = \operatorname{Nul} A^T$. Thus, we must find a basis of this null space. Reducing $\begin{bmatrix} A^T & \mathbf{0} \end{bmatrix}$ to reduced echelon form, we get

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ -1 & 8 & 2 & 0 \\ -1 & 5 & 1 & 0 \\ 0 & 9 & 3 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 + R1} \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 12 & 4 & 0 \\ -1 & 5 & 1 & 0 \\ 0 & 9 & 3 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 + R1} \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 12 & 4 & 0 \\ 0 & 9 & 3 & 0 \\ 0 & 9 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - \frac{3}{4}R2} \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 12 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 9 & 3 & 0 \end{bmatrix} \xrightarrow{R4 \to R4 - \frac{3}{4}R2} \begin{bmatrix} \boxed{1} & 4 & 2 & 0 \\ 0 & 12 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R2\to\frac{1}{12}R2}\begin{bmatrix}\boxed{1} & 4 & 2 & 0\\ 0 & 1 & \frac{1}{3} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\xrightarrow{R1\to R1-4R2}\begin{bmatrix}\boxed{1} & 0 & \frac{2}{3} & 0\\ 0 & \boxed{1} & \frac{1}{3} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\Uparrow$$

giving a general solution

$$\begin{cases} x_1 = -\frac{2}{3}x_3\\ x_2 = -\frac{1}{3}x_3\\ x_3 \text{ free}\end{cases}$$

and a vector parametric form of

$$\mathbf{x} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}-\frac{2}{3}x_3\\ -\frac{1}{3}x_3\\ x_3\end{bmatrix} = x_3\begin{bmatrix}-\frac{2}{3}\\ -\frac{1}{3}\\ 1\end{bmatrix}, \quad x_3 \text{ free}$$

Therefore, the required basis is

$$\left\{\begin{bmatrix}-2/3\\ -1/3\\ 1\end{bmatrix}\right\}$$

3. (a) By Theorem 6.13, the least-squares solutions satisfy $A^TA\mathbf{x} = A^T\mathbf{b}$. Since

$$A^TA = \begin{bmatrix}1 & 4 & 1 & 2\\ 0 & 3 & 0 & 5\\ 2 & 6 & -2 & 10\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 4 & 3 & 6\\ 1 & 0 & -2\\ 2 & 5 & 10\end{bmatrix} = \begin{bmatrix}22 & 22 & 44\\ 22 & 34 & 68\\ 44 & 68 & 144\end{bmatrix}$$

and

$$A^T\mathbf{b} = \begin{bmatrix}1 & 4 & 1 & 2\\ 0 & 3 & 0 & 5\\ 2 & 6 & -2 & 10\end{bmatrix}\begin{bmatrix}20\\ 3\\ 4\\ -7\end{bmatrix} = \begin{bmatrix}22\\ -26\\ -20\end{bmatrix}$$

we need to find a solution to

$$\begin{bmatrix}22 & 22 & 44\\ 22 & 34 & 68\\ 44 & 68 & 144\end{bmatrix}\mathbf{x} = \begin{bmatrix}22\\ -26\\ -20\end{bmatrix}$$

Reducing the augmented matrix gives:

$$\begin{bmatrix}22 & 22 & 44 & 22\\ 22 & 34 & 68 & -26\\ 44 & 68 & 144 & -20\end{bmatrix}\xrightarrow{R2\to R2-R1}\begin{bmatrix}22 & 22 & 44 & 22\\ 0 & 12 & 24 & -48\\ 44 & 68 & 144 & -20\end{bmatrix}\xrightarrow{R3\to R3-2R1}\begin{bmatrix}22 & 22 & 44 & 22\\ 0 & 12 & 24 & -48\\ 0 & 24 & 56 & -64\end{bmatrix}$$

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$$\xrightarrow{R3\to R3-2R2} \begin{bmatrix} \boxed{22} & 22 & 44 & 22 \\ 0 & \boxed{12} & 24 & -48 \\ 0 & 0 & 8 & 32 \end{bmatrix} \xrightarrow{R3\to \frac{1}{8}R3} \begin{bmatrix} \boxed{22} & 22 & 44 & 22 \\ 0 & \boxed{12} & 24 & -48 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R1\to R1-44R3} \begin{bmatrix} \boxed{22} & 22 & 0 & -154 \\ 0 & \boxed{12} & 24 & -48 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R2\to R2-24R3} \begin{bmatrix} \boxed{22} & 22 & 0 & -154 \\ 0 & 12 & 0 & -144 \\ 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

$$\xrightarrow{R2\to \frac{1}{12}R2} \begin{bmatrix} \boxed{22} & 22 & 0 & -154 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & \boxed{1} & 4 \end{bmatrix} \xrightarrow{R1\to R1-22R2} \begin{bmatrix} 22 & 0 & 0 & 110 \\ 0 & \boxed{1} & 0 & -12 \\ 0 & 0 & \boxed{1} & 4 \end{bmatrix} \xrightarrow{R1\to \frac{1}{22}R1} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & -12 \\ 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

Therefore, the unique least-squares solution is

$$\hat{\mathbf{x}} = \begin{bmatrix} 5 \\ -12 \\ 4 \end{bmatrix}$$

(b) Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$. Because the rank of A is known to be 3, all three columns form a basis for $\operatorname{Col} A$.

Applying the Gram-Schmidt process to the basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, the first vector is just the first column:

$$\mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

The second vector is calculated by:

$$\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 5 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 3 \\ 0 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 5 \end{bmatrix} - \frac{22}{22} \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$$

The third vector is

$$\mathbf{v}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 2 \\ 6 \\ -2 \\ 10 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 6 \\ -2 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 6 \\ -2 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}}{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 6 \\ -2 \\ 10 \end{bmatrix} - \frac{44}{22}\begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix} - \frac{24}{12}\begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -2 \\ 10 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

giving an orthogonal basis of

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \right\}$$

(c) We can convert the orthogonal basis to an orthonormal one by normalizing each vector. The lengths of the vectors are:

$$\begin{aligned} \|\mathbf{v}_1\| &= \sqrt{1^2 + 4^2 + 1^2 + 2^2} = \sqrt{22} \\ \|\mathbf{v}_2\| &= \sqrt{(-1)^2 + (-1)^2 + (-1)^2 + 3^2} = \sqrt{12} \\ \|\mathbf{v}_3\| &= \sqrt{2^2 + 0^2 + (-2)^2 + 0^2} = \sqrt{8} \end{aligned}$$

so the normalized vectors are

$$\mathbf{u}_1 = \frac{1}{\sqrt{22}}\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{22} \\ 4/\sqrt{22} \\ 1/\sqrt{22} \\ 2/\sqrt{22} \end{bmatrix} \qquad \mathbf{u}_2 = \frac{1}{\sqrt{12}}\mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{12} \\ -1/\sqrt{12} \\ -1/\sqrt{12} \\ 3/\sqrt{12} \end{bmatrix} \qquad \mathbf{u}_3 = \frac{1}{\sqrt{8}}\mathbf{v}_3 = \begin{bmatrix} 2/\sqrt{8} \\ 0 \\ -2/\sqrt{8} \\ 0 \end{bmatrix}$$

giving an orthnormal basis

$$\left\{ \begin{bmatrix} 1/\sqrt{22} \\ 4/\sqrt{22} \\ 1/\sqrt{22} \\ 2/\sqrt{22} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{12} \\ -1/\sqrt{12} \\ -1/\sqrt{12} \\ 3/\sqrt{12} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{8} \\ 0 \\ -2/\sqrt{8} \\ 0 \end{bmatrix} \right\}$$

(d) The orthonormal basis from part (c) gives the columns for the matrix Q:

$$Q = \begin{bmatrix} 1/\sqrt{22} & -1/\sqrt{12} & 2/\sqrt{8} \\ 4/\sqrt{22} & -1/\sqrt{12} & 0 \\ 1/\sqrt{22} & -1/\sqrt{12} & -2/\sqrt{8} \\ 2/\sqrt{22} & 3/\sqrt{12} & 0 \end{bmatrix}$$

We can calculate R using the formula $R = Q^T A$, like so:

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{22} & 4/\sqrt{22} & 1/\sqrt{22} & 2/\sqrt{22} \\ -1/\sqrt{12} & -1/\sqrt{12} & -1/\sqrt{12} & 3/\sqrt{12} \\ 2/\sqrt{8} & 0 & -2/\sqrt{8} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 4 & 3 & 6 \\ 1 & 0 & -2 \\ 2 & 5 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 22/\sqrt{22} & 22/\sqrt{22} & 44/\sqrt{22} \\ 0 & 12/\sqrt{12} & 24/\sqrt{12} \\ 0 & 0 & 8/\sqrt{8} \end{bmatrix} = \begin{bmatrix} \sqrt{22} & \sqrt{22} & 2\sqrt{22} \\ 0 & \sqrt{12} & 2\sqrt{12} \\ 0 & 0 & \sqrt{8} \end{bmatrix}$$

Ugly, but not impossible.

4. (a) The eigenvalues of A are given by the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 8-\lambda & -4 & 0 \\ -4 & 2-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda) \begin{vmatrix} 8-\lambda & -4 \\ -4 & 2-\lambda \end{vmatrix}$$

$$= (5-\lambda)((8-\lambda)(2-\lambda) - 16) = (5-\lambda)(\lambda^2 - 10\lambda) = (5-\lambda)\lambda(\lambda - 10)$$

giving eigenvalues 0, 5, and 10.

For $\lambda = 0$, the eigenspace is given by the solution set associated with augmented matrix

$$\begin{bmatrix} A - 0I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

Reducing gives

$$\begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 + \frac{1}{2}R1} \begin{bmatrix} 8 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} \boxed{8} & -4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R2 \to \frac{1}{5}R2} \begin{bmatrix} 8 & -4 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to \frac{1}{8}R1} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is

$$\begin{cases} x_1 = \frac{1}{2}x_2 \\ x_3 = 0 \\ x_2 \text{ free} \end{cases}$$

and the vector parametric form is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 \text{ free}$$

giving eigenspace basis

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

For $\lambda = 5$, the eigenspace is given by the reduction

$$\begin{bmatrix} A - 5I & \mathbf{0} \end{bmatrix} \begin{bmatrix} 3 & -4 & 0 & 0 \\ -4 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 + \frac{4}{3}R1} \begin{bmatrix} \boxed{3} & -4 & 0 & 0 \\ 0 & -\frac{25}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R2 \to -\frac{3}{25}R2} \begin{bmatrix} \boxed{3} & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 + 4R2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to \frac{1}{3}R1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 \text{ free} \end{cases}$$

and the vector parametric form is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 \text{ free}$$

giving eigenspace basis

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Finally, for $\lambda = 10$, the eigenspace is given by the reduction

$$\begin{bmatrix} A - 10I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -2 & -4 & 0 & 0 \\ -4 & -8 & 0 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} -2 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} \boxed{-2} & -4 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \to -\frac{1}{5}R2} \begin{bmatrix} -2 & -4 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1 \to -\frac{1}{2}R1} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is

$$\begin{cases} x_1 = -2x_2 \\ x_3 = 0 \\ x_2 \text{ free} \end{cases}$$

and the vector parametric form is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 \text{ free}$$

giving eigenspace basis

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Since A is symmetric, these three basis vectors, each from a different eigenspace, must be orthogonal, so there is no need to orthogonalize anything. However, the vectors must still be normalized:

$$(\lambda = 0) \qquad \frac{1}{\left\| \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \right\|} \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5/4}} \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = \frac{2}{\sqrt{5}} \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$$

$$(\lambda = 5) \qquad \frac{1}{\left\| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\|} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{1}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(\lambda = 10) \qquad \frac{1}{\left\| \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\|} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

This gives matrices

$$P = \begin{bmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

(b) Writing $P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$, the spectral decomposition of A is:

$$\begin{aligned} A &= \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \lambda_3 \mathbf{u}_3 \mathbf{u}_3^T \\ &= 0 \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} + 10 \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix} \\ &= 0 \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 5 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 10 \begin{bmatrix} 4/5 & -2/5 & 0 \\ -2/5 & 1/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(c) Because A's eigenvalues are all greater than or equal to zero, it is positive semidefinite (but *not* positive definite, since one of the eigenvalues is 0).

5. (a) Because A is lower triangular, its eigenvalues are the entries on its main diagonal, namely 1 and 4. For $\lambda = 1$, the eigenspace is given by

$$\begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} \xrightarrow[R2 \to R2 - 3R1]{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

giving vector parametric form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_2 \text{ free}$$

and the basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

For $\lambda = 4$, the eigenspace is given by

$$\begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

giving vector parametric form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and the basis

$$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

This gives matrices

$$P = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

(b) An eigenvector for eigenvalue 1 is given by the basis vector $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ above. Similarly, an eigenvector for eigenvalue 4 is $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

If the initial colony is given by $\mathbf{x}_0 = c\mathbf{u}$, then

$$\mathbf{x}_1 = A\mathbf{x}_0 = A(c\mathbf{u}) = cA\mathbf{u} = c\mathbf{u} = \mathbf{x}_0$$

with the second-last equality following from the fact that $\mathbf{u}$ is an eigenvector for eigenvalue 1. Similarly, $\mathbf{x}_2 = \mathbf{x}_0$, and so on. That is, a population started at a scalar multiple of $\mathbf{u}$ does not change from hour to hour.

On the other hand, if the initial colony is given by $\mathbf{x}_0 = c\mathbf{v}$, then

$$\mathbf{x}_1 = A\mathbf{x}_0 = A(c\mathbf{v}) = cA\mathbf{v} = 4c\mathbf{v} = 4\mathbf{x}_0$$

with the second-last equality following from the fact that $\mathbf{v}$ is an eigenvector for eigenvalue 4. Similarly,

$$\mathbf{x}_2 = A\mathbf{x}_1 = A(4c\mathbf{v}) = 4cA\mathbf{v} = 4^2c\mathbf{v} = 4^2\mathbf{x}_0$$

That is, a population started at a scalar multiple of $\mathbf{v}$ quadruples in size every hour without the proportions of wild type and heat sensitive bacteria changing.

(c) Since $\mathbf{u}$ and $\mathbf{v}$ are not scalar multiples of each other, they are two, linearly independent vectors in $\mathbb{R}^2$, so they must form a basis for $\mathbb{R}^2$.

(d) This follows from the same argument as in part (b). That is,

$$\mathbf{x}_1 = A\mathbf{x}_0 = A(c\mathbf{u} + d\mathbf{v}) = cA\mathbf{u} + dA\mathbf{v} = c\mathbf{u} + 4d\mathbf{v}$$

with the second-last equality following from the linearity of $\mathbf{x} \mapsto A\mathbf{x}$ and the final equality following from the fact that $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for eigenvalues 1 and 4 respectively.

Similarly,

$$\mathbf{x}_2 = A\mathbf{x}_1 = A(c\mathbf{u} + 4d\mathbf{v}) = cA\mathbf{u} + 4dA\mathbf{v} = c\mathbf{u} + 4^2d\mathbf{v}$$

The appropriate formula for general k will be:

$$\mathbf{x}_k = c\mathbf{u} + 4^k d\mathbf{v}$$

(e) The vector $\mathbf{x}_0 = (10, 0)$ can be expressed as

$$\mathbf{x}_0 = \begin{bmatrix} 10 \\ 0 \end{bmatrix} = -\frac{10}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{10}{3}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = -\frac{10}{3}\mathbf{u} + \frac{10}{3}\mathbf{v}$$

These weights $c = -10/3$ and $d = 10/3$ are calculated in the usual manner by solving the vector equation $\mathbf{x}_0 = c\mathbf{u} + d\mathbf{v}$ for unknowns c and d.

By part (d), we have

$$\mathbf{x}_k = c\mathbf{u} + 4^k d\mathbf{v} = -\frac{10}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 4^k\frac{10}{3}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 10(4^k) \\ (10/3)(4^k - 1) \end{bmatrix}$$

which gives

$$\mathbf{x}_3 = \begin{bmatrix} 10(4^3) \\ (10/3)(4^3 - 1) \end{bmatrix} = \begin{bmatrix} 10(64) \\ (10/3)(63) \end{bmatrix} = \begin{bmatrix} 640 \\ 210 \end{bmatrix}$$

If you missed the correction to the quiz and used $\mathbf{x}_0 = (1, 0)$ instead, you should get the formula

$$\mathbf{x}_k = \begin{bmatrix} 4^k \\ (4^k - 1)/3 \end{bmatrix}$$

and

$$\mathbf{x}_3 = \begin{bmatrix} 4^3 \\ (4^3 - 1)/3 \end{bmatrix} = \begin{bmatrix} 64 \\ 21 \end{bmatrix}$$