Math 221 (101) Matrix Algebra November 13, 2002

Quiz #8

DUE: Wednesday, November 20 at 9:00am

Rules:

1. Quizzes are due at the beginning of class on the due date. Late quizzes will not receive credit. Remember to include your name and student number on your answers.
2. You may work with other Math 221 students and use materials such as textbooks, notes, calculators, or computers. However, you must write up your own original, complete solutions. Copying all or part of an answer is not allowed.
3. Show all work. Explain your solutions clearly. You may write in pencil if you'd like.
4. The maximum possible score is 20.

Solve the following five problems, showing all work. Don't forget to put your name and student number on your answers.

1. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

(a) Calculate $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_1 \cdot \mathbf{v}_3$, and $\mathbf{v}_2 \cdot \mathbf{v}_3$. Which of these vectors, if any, are orthogonal?

(b) Calculate $\|\mathbf{v}_1\|$, $\|\mathbf{v}_2\|$, and $\|\mathbf{v}_3\|$.

(c) Calculate the distance between $\mathbf{v}_1$ and $\mathbf{v}_2$.

2. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. For the following vectors,

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{p}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

suppose that $T(\mathbf{p}_1) = \mathbf{p}_1$, $T(\mathbf{p}_2) = 2\mathbf{p}_2$, and $T(\mathbf{p}_3) = 2\mathbf{p}_3$.

(a) Calculate the standard matrix A for T. [Hint: Use the Diagonalization Theorem.]

(b) Find the matrix for T relative to the basis $\mathfrak{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$. [Hint: No further calculation is necessary.]

3. Consider the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

(a) Show that these two vectors are orthogonal.

(b) Let $\mathbf{v} = (-1, -8, -13)$. Find the orthogonal projection $\hat{\mathbf{v}}$ of $\mathbf{v}$ onto $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Calculate $\mathbf{z} = \mathbf{v} - \hat{\mathbf{v}}$.

(c) Explain why $\mathfrak{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{z}\}$ forms an orthogonal basis for $\mathbb{R}^3$.

(d) Find the coordinate vector of $(1, 0, -1)$ with respect to $\mathfrak{B}$. [Hint: You may wish to use Theorem 6.5.]

4. Let U be an $m \times n$ matrix. Prove that the columns of the matrix are orthogonal iff $U^T U$ is a diagonal matrix.

5. Let $\mathbf{u} \neq \mathbf{0}$ be a point in $\mathbb{R}^n$. We will show that the set of all points that are equidistant from $\mathbf{u}$ and $-\mathbf{u}$ (that is, the set of all points $\mathbf{x}$ with $\text{dist}(\mathbf{x}, \mathbf{u}) = \text{dist}(\mathbf{x}, -\mathbf{u})$) is the orthogonal complement of the line $\text{Span}\{\mathbf{u}\}$.

Let W be the set of all points equidistant from $\mathbf{u}$ and $-\mathbf{u}$.

(a) Explain why $\mathbf{x} \in W$ iff $\|\mathbf{x} - \mathbf{u}\| = \|\mathbf{x} + \mathbf{u}\|$ iff $(\mathbf{x} - \mathbf{u}) \cdot (\mathbf{x} - \mathbf{u}) = (\mathbf{x} + \mathbf{u}) \cdot (\mathbf{x} + \mathbf{u})$.

(b) Use the properties of the inner product given in Theorem 6.1 to show that

$$(\mathbf{x} - \mathbf{u}) \cdot (\mathbf{x} - \mathbf{u}) = \mathbf{x} \cdot \mathbf{x} - 2\mathbf{x} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}$$

and

$$(\mathbf{x} + \mathbf{u}) \cdot (\mathbf{x} + \mathbf{u}) = \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}$$

(c) Using parts (a) and (b), show that $\mathbf{x} \in W$ iff $\mathbf{x} \cdot \mathbf{u} = 0$.

(d) Explain why part (c) implies that $\mathbf{x} \in W$ iff $\mathbf{x} \cdot \mathbf{v} = 0$ for *every* vector $\mathbf{v} \in \text{Span}\{\mathbf{u}\}$ (that is, for every scalar multiple $\mathbf{v} = c\mathbf{u}$).

Note that part (d), by definition, says that $W = (\text{Span}\{\mathbf{u}\})^\perp$.