Quiz #8 Solutions

1. (a)

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = 1(4) + 2(-1) + 3(1) = 5$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = 1(-3) + 2(0) + 3(1) = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = 4(-3) + (-1)(0) + 1(1) = -11$$

Only $\mathbf{v}_1$ and $\mathbf{v}_3$ are an orthogonal pair.

(b)

$$\|\mathbf{v}_1\| = \sqrt{\mathbf{v}_1 \cdot \mathbf{v}_1} = \sqrt{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\|\mathbf{v}_2\| = \sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2} = \sqrt{\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}} = \sqrt{4^2 + (-1)^2 + 1^2} = \sqrt{18}$$

$$\|\mathbf{v}_3\| = \sqrt{\mathbf{v}_3 \cdot \mathbf{v}_3} = \sqrt{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}} = \sqrt{(-3)^2 + 0^2 + 1^2} = \sqrt{10}$$

(c)

$$\operatorname{dist}(\mathbf{v}_1, \mathbf{v}_2) = \|\mathbf{v}_1 - \mathbf{v}_2\| = \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix} \right\| = \sqrt{(-3)^2 + 3^2 + 2^2} = \sqrt{22}$$

2. (a) From the given information, $A\mathbf{p}_1 = \mathbf{p}_1$, $A\mathbf{p}_2 = 2\mathbf{p}_2$, and $A\mathbf{p}_3 = 2\mathbf{p}_3$. That is, by definition, $\mathbf{p}_1$ is an eigenvector for A associated with eigenvalue 1, and $\mathbf{p}_2$ and $\mathbf{p}_3$ are eigenvectors for A associated with eigenvalue 2. Note that $\mathbf{p}_2$ and $\mathbf{p}_3$ are linearly independent since neither is a scalar multiple of the other. In short, we have three linearly independent eigenvectors and their associated eigenvalues.

By the Diagonalization Theorem, we may write

$$A = PDP^{-1}$$

for an invertible matrix P whose columns are these eigenvectors:

$$P = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

and a diagonal matrix D whose diagonal entries are the corresponding eigenvalues:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

We know P and D, so to calculate the matrix A, we need only calculate P^{-1}. Using the usual algorithm:

$$\begin{bmatrix} P & I \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ -2 & 1 & -2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 + 2R1} \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 7 & 2 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - R1} \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 7 & 2 & 2 & 1 & 0 \\ 0 & -3 & -1 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{R3 \to R3 + \frac{3}{7} R2} \begin{bmatrix} \boxed{1} & 3 & 2 & 1 & 0 & 0 \\ 0 & \boxed{7} & 2 & 2 & 1 & 0 \\ 0 & 0 & -\frac{1}{7} & -\frac{1}{7} & \frac{3}{7} & 1 \end{bmatrix}$$

$$\xrightarrow{R3 \to -7R3} \begin{bmatrix} \boxed{1} & 3 & 2 & 1 & 0 & 0 \\ 0 & \boxed{7} & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3 & -7 \end{bmatrix} \xrightarrow{R1 \to R1 - 2R3} \begin{bmatrix} \boxed{1} & 3 & 0 & -1 & 6 & 14 \\ 0 & \boxed{7} & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3 & -7 \end{bmatrix}$$

$$\xrightarrow{R2 \to R2 - 2R3} \begin{bmatrix} \boxed{1} & 3 & 0 & -1 & 6 & 14 \\ 0 & 7 & 0 & 0 & 7 & 14 \\ 0 & 0 & \boxed{1} & 1 & -3 & -7 \end{bmatrix} \xrightarrow{R2 \to \frac{1}{7} R2} \begin{bmatrix} \boxed{1} & 3 & 0 & -1 & 6 & 14 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & \boxed{1} & 1 & -3 & -7 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 - 3R2} \begin{bmatrix} \boxed{1} & 0 & 0 & -1 & 3 & 8 \\ 0 & \boxed{1} & 0 & 0 & 1 & 2 \\ 0 & 0 & \boxed{1} & 1 & -3 & -7 \end{bmatrix} = \begin{bmatrix} I & P^{-1} \end{bmatrix}$$

That is

$$P^{-1} = \begin{bmatrix} -1 & 3 & 8 \\ 0 & 1 & 2 \\ 1 & -3 & -7 \end{bmatrix}$$

so

$$A = PDP^{-1} = \left(\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} -1 & 3 & 8 \\ 0 & 1 & 2 \\ 1 & -3 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 4 \\ -2 & 2 & -4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 & 8 \\ 0 & 1 & 2 \\ 1 & -3 & -7 \end{bmatrix} = \begin{bmatrix} 3 & -3 & -8 \\ -2 & 8 & 16 \\ 1 & -3 & -6 \end{bmatrix}$$

(b) By Theorem 5.8 (p. 325), this is just the diagonal matrix D:

$$[T]_{\mathfrak{B}} = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

3. (a) $\mathbf{u}_1 \cdot \mathbf{u}_2 = 1(2) + (-2)(1) + 0(-4) = 0$

(b) By the Orthogonal Decomposition Theorem,

$$\hat{\mathbf{v}} = \left(\frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{v}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\right)\mathbf{u}_2 = \frac{\begin{bmatrix} -1 \\ -8 \\ -13 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} -1 \\ -8 \\ -13 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

$$= \frac{15}{5}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \frac{42}{21}\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ -8 \end{bmatrix}$$

And

$$\mathbf{z} = \mathbf{v} - \hat{\mathbf{v}} = \begin{bmatrix} -1 \\ -8 \\ -13 \end{bmatrix} - \begin{bmatrix} 7 \\ -4 \\ -8 \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \\ -5 \end{bmatrix}$$

Note that we can check our work by verifying that $\mathbf{z}$ is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$ (which it is).

(c) By the Orthogonal Decomposition Theorem (or the way we checked our work), we know that $\mathbf{z}$ is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. In part (a), we showed that $\mathbf{u}_1 \perp \mathbf{u}_2$. Therefore, all pairs of these vectors are orthogonal, so $\mathfrak{B}$ is an orthogonal set.

However, by Theorem 6.4, $\mathfrak{B}$ orthogonal and all nonzero implies $\mathfrak{B}$ is linearly independent. Any set of the three linearly independent vectors is a basis for $\mathbb{R}^3$ by the Basis Theorem.

Therefore, $\mathfrak{B}$ is a basis for $\mathbb{R}^3$ that is also an orthogonal set, so it is an orthogonal basis for $\mathbb{R}^3$.

(d) Using Theorem 6.5,

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} -8 \\ -4 \\ -5 \end{bmatrix}$$

where

$$c_1 = \frac{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}} = \frac{1}{5} \qquad c_2 = \frac{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}} = \frac{6}{21} \qquad c_3 = \frac{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ -4 \\ -5 \end{bmatrix}}{\begin{bmatrix} -8 \\ -4 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ -4 \\ -5 \end{bmatrix}} = \frac{-3}{105}$$

giving a coordinate vector

$$\begin{bmatrix} -8 \\ -4 \\ -5 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 1/5 \\ 6/21 \\ -3/105 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/7 \\ -1/35 \end{bmatrix}$$

4. Let the entries of the $n \times n$ matrix U^TU be $U^TU = [c_{ij}]$. By the Row-Column Rule for Computing AB on p. 103, the (i,j)th entry of U^TU may be written

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \tag{1}$$

where $U^T = [a_{ij}]$ and $U = [b_{ij}]$. Observe that $a_{ij} = b_{ji}$ by the definition of transposition.

Now, as $U = \begin{bmatrix} \mathbf{u}_1 & \ldots & \mathbf{u}_n \end{bmatrix}$ are the columns of U, we have

$$\mathbf{u}_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

and, furthermore,

$$\mathbf{u}_i = \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$$

so equation (1) can be rewritten

$$c_{ij} = \mathbf{u}_i \cdot \mathbf{u}_j$$

That is, the (i,j)th column of U^TU is the inner product $\mathbf{u}_i \cdot \mathbf{u}_j$.

If U has orthogonal columns, then—by definition—we have

$$c_{ij} = \mathbf{u}_i \cdot \mathbf{u}_j = 0$$

for all $i \neq j$. That is, all off-diagonal elements of U are zero, so U is a diagonal matrix.

Conversely, if U is a diagonal matrix, then all its off-diagonal elements are zero, implying

$$\mathbf{u}_i \cdot \mathbf{u}_j = c_{ij} = 0$$

for all $i \neq j$. But this means the columns of U are orthogonal.

That's all we needed to prove.

5. (a) By definition $\mathbf{x} \in W$ iff $\operatorname{dist}(\mathbf{x}, \mathbf{u}) = \operatorname{dist}(\mathbf{x}, -\mathbf{u})$ iff $\|\mathbf{x} - \mathbf{u}\| = \|\mathbf{x} - (-\mathbf{u})\|$ iff $\|\mathbf{x} - \mathbf{u}\| = \|\mathbf{x} + \mathbf{u}\|$ iff $\sqrt{(\mathbf{x} - \mathbf{u}) \cdot (\mathbf{x} - \mathbf{u})} = \sqrt{(\mathbf{x} + \mathbf{u}) \cdot (\mathbf{x} + \mathbf{u})}$. But these are equal if and only if their squares are equal: $(\mathbf{x} - \mathbf{u}) \cdot (\mathbf{x} - \mathbf{u}) = (\mathbf{x} + \mathbf{u}) \cdot (\mathbf{x} + \mathbf{u})$.

(b) By the properties of inner products,

$$\begin{aligned}
(\mathbf{x} - \mathbf{u}) \cdot (\mathbf{x} - \mathbf{u}) &= (\mathbf{x} - \mathbf{u}) \cdot \mathbf{x} + (\mathbf{x} - \mathbf{u}) \cdot (-\mathbf{u}) \\
&= \mathbf{x} \cdot \mathbf{x} - \mathbf{u} \cdot \mathbf{x} + \mathbf{x} \cdot (-\mathbf{u}) - \mathbf{u} \cdot (-\mathbf{u}) \\
&= \mathbf{x} \cdot \mathbf{x} - \mathbf{u} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} \\
&= \mathbf{x} \cdot \mathbf{x} - 2\mathbf{x} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\mathbf{x} + \mathbf{u}) \cdot (\mathbf{x} + \mathbf{u}) &= (\mathbf{x} + \mathbf{u}) \cdot \mathbf{x} + (\mathbf{x} + \mathbf{u}) \cdot \mathbf{u} \\
&= \mathbf{x} \cdot \mathbf{x} + \mathbf{u} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} \\
&= \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}
\end{aligned}$$

(c) By part (a), $x \in W$ iff $(\mathbf{x} - \mathbf{u}) \cdot (\mathbf{x} - \mathbf{u}) = (\mathbf{x} + \mathbf{u}) \cdot (\mathbf{x} + \mathbf{u})$. But, by part (b), this is true iff

$$\mathbf{x} \cdot \mathbf{x} - 2\mathbf{x} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}$$

Subtracting $\mathbf{x} \cdot \mathbf{x}$ and $\mathbf{u} \cdot \mathbf{u}$ from both sides, this is true iff

$$\begin{aligned}
-2\mathbf{x} \cdot \mathbf{u} &= 2\mathbf{x} \cdot \mathbf{u} \\
-4\mathbf{x} \cdot \mathbf{u} &= 0 \\
\mathbf{x} \cdot \mathbf{u} &= 0
\end{aligned}$$

That is, $\mathbf{x} \in W$ iff $\mathbf{x} \cdot \mathbf{u} = 0$.

(d) We need only prove that $\mathbf{x} \cdot \mathbf{u} = 0$ iff $\mathbf{x} \cdot \mathbf{v} = 0$ for every $\mathbf{v} = c\mathbf{u}$. This fact, combined with part (c), will give the result.

If $\mathbf{x} \cdot \mathbf{u} = 0$, then

$$\mathbf{x} \cdot \mathbf{v} = \mathbf{x} \cdot (c\mathbf{u}) = c\mathbf{x} \cdot \mathbf{u} = c0 = 0$$

Conversely, if $\mathbf{x} \cdot \mathbf{v} = 0$ for every $\mathbf{v} = c\mathbf{u}$ then it's true, in particular, for $c = 1$, and $\mathbf{x} \cdot \mathbf{u} = 0$.