# Quiz #7

**DUE: Wednesday, November 6 at 9:00am**

**Rules:**

1. Quizzes are due at the beginning of class on the due date. Late quizzes will not receive credit. Remember to include your name and student number on your answers.
2. You may work with other Math 221 students and use materials such as textbooks, notes, calculators, or computers. However, you must write up your own original, complete solutions. Copying all or part of an answer is not allowed.
3. Show all work. Explain your solutions clearly. You may write in pencil if you'd like.
4. The maximum possible score is 20.

Solve the following five problems, showing all work. Don't forget to put your name and student number on your answers.

**1.** Let $A$ be a $3 \times 4$ matrix of rank 2.

(a) What is the dimension of $\operatorname{Nul} A$?

(b) If the following four vectors are known to be solutions of $A\mathbf{x} = \mathbf{0}$, give a basis for $\operatorname{Nul} A$.

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_3 = \begin{bmatrix} -3 \\ -2 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{p}_4 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

[Hint: The Basis Theorem should help.]

**2.** Consider the system of equations

$$\begin{cases} x_1 + x_3 + x_4 + 2x_5 = 0 & (1) \\ x_2 + x_4 + 3x_5 = 0 & (2) \end{cases}$$

(a) Write down the $2 \times 5$ coefficient matrix $A$. Find a basis for $\operatorname{Nul} A$, and determine its dimension.

(b) Suppose we add the equation

$$2x_1 - x_2 + 2x_3 + 2x_4 + x_5 = 0 \qquad (3)$$

to the system $(1), (2)$. Write down the $3 \times 5$ coefficient matrix $B$ for this new system $(1), (2), (3)$. Find a basis for $\operatorname{Nul} B$, and determine its dimension.

**3.** Let $A$ be the triangular matrix given by

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(a) Calculate the determinant of $A$.

(b) Find the inverse of $A$.

(c) What are the eigenvalues of $A$? What are the eigenvalues of $A^{-1}$?

(d) Find a basis for the eigenspace associated with $A$'s *smallest* eigenvalue.

**4.** Let the matrix $A$ be given by

$$A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$$

(a) Find the two eigenvalues of $A$.

(b) Find a basis for each of the two eigenspaces associated with these eigenvalues.

(c) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

(d) Calculate $P^{-1}$, and check that $A = PDP^{-1}$.

(e) Calculate $A^{10}$. [Note: $2^{10} = 1024$.]

**5.** (a) Let $A$ and $B$ be $n \times n$ matrices. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ such that $\mathbf{v}$ is an eigenvector of $A$ with eigenvalue 2 (that is, $A\mathbf{v} = 2\mathbf{v}$) and $\mathbf{v}$ is also an eigenvector of $B$ with eigenvalue 3 (that is, $B\mathbf{v} = 3\mathbf{v}$). Show that $\mathbf{v}$ is an eigenvector of $AB$ with eigenvalue 6.

(b) Let $C$ and $D$ be $2 \times 2$ matrices given by

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

Show that 2 is an eigenvalue of $C$ and 3 is an eigenvalue of $D$, but 6 is not an eigenvalue of $CD$. [Hint: Use the characteristic equation.]