Math 221 (101) Matrix Algebra October 23, 2002

Quiz #6

DUE: Wednesday, October 30 at 9:00am

Rules:

1. Quizzes are due at the beginning of class on the due date. Late quizzes will not receive credit. Remember to include your name and student number on your answers.
2. You may work with other Math 221 students and use materials such as textbooks, notes, calculators, or computers. However, you must write up your own original, complete solutions. Copying all or part of an answer is not allowed.
3. Show all work. Explain your solutions clearly. You may write in pencil if you'd like.
4. The maximum possible score is 20.

Solve the following five problems, showing all work. Don't forget to put your name and student number on your answers.

1. Let the matrix A be given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

(a) Give an example of a (nonzero) vector in $\operatorname{Col} A$. Is the vector $(4, 11)$ in $\operatorname{Col} A$? Give a basis for $\operatorname{Col} A$.

(b) Express $\operatorname{Nul} A$ in vector parametric form. Give a basis for $\operatorname{Nul} A$. Give an example of a (nonzero) vector in $\operatorname{Nul} A$. Is the vector $(1, 2, 1)$ in $\operatorname{Nul} A$?

(c) Calculate the dimensions of $\operatorname{Col} A$ and $\operatorname{Nul} A$ and verify that the Rank Theorem holds for this matrix.

2. Let H be a subspace of $\mathbb{R}^n$ with basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

(a) What is the dimension of H?

(b) Does every basis for H contains three vectors? Why or why not? Is every set of three nonzero vectors in H a basis for H? Why or why not?

(c) Is $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 - \mathbf{b}_1\}$ a basis for H? Why or why not? What about $\{\mathbf{b}_1, \mathbf{b}_2 + \mathbf{b}_3\}$?

3. Let R be a triangle in $\mathbb{R}^2$ with corners at points $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$.

(a) Consider the linear transformation T that maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Calculate the standard matrix A for this transformation.

(b) Note that the transformation T transforms the triangle S with corners $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to the triangle $R = T(S)$ with corners $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$. Using the boxed fact on p. 203 (the generalization of Theorem 3.10 to general regions), give a formula for the area of the triangle R in terms of the elements of $\mathbf{u}$ and $\mathbf{v}$.

4. In the text, the Matrix Inverse Formula (Theorem 3.8, p. 198) is proved using Cramer's Rule. Let's try a more direct proof.

Let A be an $n \times n$, invertible matrix, and define

$$B = \frac{1}{\det A} D$$

where $D = [d_{ij}]$ is the adjugate of A having elements $d_{ij} = C_{ji}$. We will show that $BA = I$. Then, the Matrix Inverse Formula will follow from the boxed fact on p. 121.

To show that $BA = I$:

(a) Use the boxed "Row-Column Rule for Computing AB" on p. 103 to write down an expression for the ith element of the main diagonal (the element in the ith row and ith column) of DA. Explain why this expression is equal to $\det A$. Note that this implies that the diagonal elements of BA are all ones.

(b) Show that for any matrix F with two identical columns we have $\det F = 0$. [Hint: Are the columns of F linearly dependent or independent? So, is F singular or nonsingular?]

(c) For any i and j with $i \neq j$, let F be the matrix that looks like A except its ith column has been replaced by a copy of its jth column (so its ith and jth columns are the same, both equal to the original jth column of A). Explain why the element in the ith row and jth column of DA is equal to $\det F$. Note that, by part (b), we have $\det F = 0$, so the off-diagonal elements of DA, and so BA, are all zeros.

We have now shown that $BA = I$, and the proof is complete.

5. Let $A_1, A_2, \ldots, A_p$ all be $n \times n$ square matrices. Prove that their product $A_1 A_2 \ldots A_p$ is invertible iff all of the individual matrices are invertible. [Hint: Use determinants.]