# Quiz #6 Solutions

**1.** (a) Any (nonzero) linear combination of the columns of $A$ will do. The simplest examples are the values of the columns themselves: $(1,4)$, $(2,5)$, or $(3,6)$. (In fact, because the columns of $A$ span $\mathbb{R}^2$ as shown below, *any* nonzero vector in $\mathbb{R}^2$ will work, but you'd have to justify this to get full marks.)

Yes, the vector $(4,11)$ is in $\operatorname{Col} A$. We can see this directly by reducing the augmented matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 11 \end{bmatrix} \xrightarrow{R2 \to R2 - 4R1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -5 \end{bmatrix}$$

This is clearly consistent, so the given vector is a linear combination of the columns of $A$ and so is in $\operatorname{Col} A$. (Alternatively, if we reduced the coefficient matrix itself, we would discover that it has a pivot position in every row. Thus, $A$ spans all of $\mathbb{R}^2$ and *every* vector in $\mathbb{R}^2$ is in $\operatorname{Col} A$.)

The usual solution would involve row reducing $A$ to echelon form to find its pivot columns:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow{R2 \to R2 - 4R1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix}$$

Thus, the first two columns of $A$ are its pivot columns, so

$$\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$$

forms a basis for $\operatorname{Col} A$.

(In fact, because $\operatorname{Col} A = \mathbb{R}^2$, *any* two linearly independent vectors in $\mathbb{R}^2$ form a basis for $\mathbb{R}^2$. Even the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ would work. Again, though, you'd have to justify your answer to get full marks.)

(b) Solving the homogeneous system:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 - 4R1} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

$$\xrightarrow{R2 \to -\frac{1}{3}R2} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - 2R2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

gives the general solution

$$\begin{cases} x_1 = x_3 \\ x_2 = -2x_3 \\ x_3 \text{ free} \end{cases}$$

which we may write in vector parametric form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad x_3 \text{ free}$$

Therefore, a basis for $\operatorname{Nul} A$ is given by:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

That is, $\operatorname{Nul} A$ consists of all the scalar multiples of $(1, -2, 1)$. Therefore, any (nonzero) scalar multiple of $(1, -2, 1)$ will work as an example, such as $(1, -2, 1)$ or $(2, -4, 2)$.

The vector $(1, 2, 1)$ is *not* a scalar multiple of $(1, -2, 1)$, so it is not in $\operatorname{Nul} A$. Alternatively, you could note that

$$A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so it is *not* a solution to the homogeneous equation and, therefore, not in $\operatorname{Nul} A$.

(c) From part (a), $\operatorname{Col} A$ has a basis of two vectors, so its dimension is 2. From part (b), $\operatorname{Nul} A$ has a basis of one vector, so its dimension is 1. As we would expect from the Rank Theorem, the sum of these two numbers is 3, the number of columns in $A$.

**2.** (a) As $H$ has a basis consisting of three vectors, its dimension is 3.

(b) Yes, every basis for $H$ contains three vectors. We proved in class that all bases of a subspace have the same number of vectors. That's how we define the dimension.

No. Every set of three *linearly independent* vectors in $H$ is a basis for $H$ (by the Basis Theorem), but not every set of nonzero vectors. If the vectors were all scalar multiples of each other, for example, they would be linearly dependent and not a basis.

(c) Yes, $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 - \mathbf{b}_1\}$ is a basis for $H$. By the Basis Theorem, it suffices to show that either this set spans $H$ or that it is linearly independent.

To see that it spans $H$, note that $\mathbf{b}_3$ can be written as a linear combination

$$\mathbf{b}_3 = \mathbf{b}_1 + (\mathbf{b}_3 - \mathbf{b}_1)$$

of $\mathbf{b}_1$ and $\mathbf{b}_3 - \mathbf{b}_1$. Therefore, any vector $\mathbf{v}$ in $H$ can be rewritten:

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 \\ &= c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3(\mathbf{b}_1 + (\mathbf{b}_3 - \mathbf{b}_1)) \\ &= (c_1 + c_3)\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3(\mathbf{b}_3 - \mathbf{b}_1) \end{aligned}$$

as a linear combination of these new vectors. Therefore, by the Basis Theorem, they form a basis for $H$

Alternatively, to see that the set is linearly independent, note that if we had:

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3(\mathbf{b}_3 - \mathbf{b}_1) = 0$$

then that equation can be rewritten

$$(c_1 - c_3)\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = 0$$

By the linear independence of the original basis, we have all weights zero: $c_1 - c_3 = 0$, $c_2 = 0$, and $c_3 = 0$. But, this implies $c_1 = c_2 = c_3 = 0$, so these new vectors are linearly independent, too, and—by the Basis Theorem—they form a basis for $H$.

On the other hand, $\{\mathbf{b}_1, \mathbf{b}_2 + \mathbf{b}_3\}$ can't form a basis for $H$ because it only contains two vectors, and all bases of $H$ must have three vectors. (In terms of the definition of a basis, it turns out that this set is linearly independent, but it does not span all of $H$.)

**3.** (a) The standard matrix is given by

$$A = \begin{bmatrix} T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

(b) The area of triangle $S$ is $1/2$, so the area of the transformed triangle $R$ is given by the formula:

$$\begin{aligned} \operatorname{area}(R) &= |\det A| \operatorname{area}(S) \\ &= \frac{|u_1 v_2 - u_2 v_1|}{2} \end{aligned}$$

**4.** (a) Fix some $i$. The $i$th element of the main diagonal of $DA$ is given by

$$\begin{aligned} (DA)_{ii} &= d_{i1}a_{1i} + d_{i2}a_{2i} + \cdots + d_{in}a_{ni} \\ &= C_{1i}a_{1i} + C_{2i}a_{2i} + \cdots + C_{ni}a_{ni} \end{aligned}$$

This is actually a cofactor expansion down the $i$th column of $A$, so by Theorem 3.1, it is equal to $\det A$. Therefore, the diagonal elements of $BA = \frac{1}{\det A} DA$ are all ones.

(b) Let $F$ be such a matrix. Because it has two equal columns, its columns are linearly dependent (since one of the duplicates can be written as a linear combination of the other columns). By the Invertible Matrix Theorem, $F$ is not invertible, so by Theorem 3.4, it has $\det F = 0$.

(c) The element in the $i$th row and $j$th column of $DA$ is given by

$$\begin{aligned} (DA)_{ij} &= d_{i1}a_{1j} + d_{i2}a_{2j} + \cdots + d_{in}a_{nj} \\ &= C_{1i}a_{1j} + C_{2i}a_{2j} + \cdots + C_{ni}a_{nj} \end{aligned}$$

If we performed a cofactor expansion of $F$ down its $i$th column, we would find that the cofactors of $F$ down this column are the same as the corresponding cofactors of $A$; that is, they are $C_{1i}, \ldots, C_{ni}$. (This is because $F$ looks like $A$ except for the $i$th column and all these cofactors have the $i$th column deleted.) Therefore, by Theorem 3.1, we would have

$$\det F = f_{1i}C_{1i} + f_{2i}C_{2i} + \cdots + f_{ni}C_{ni}$$

However, the elements $f_{1i}, \ldots, f_{ni}$ of the $i$th column of $F$ are actually the elements of the $j$th column of $A$. That is,

$$\det F = a_{1j}C_{1i} + a_{2j}C_{2i} + \cdots + a_{nj}C_{ni}$$

which is the same as the expression we got for $(DA)_{ij}$ above.

By part (b), the determinant of $F$ is zero, so

$$(DA)_{ij} = \det F = 0$$

for all $i \neq j$. As a result, $(BA)_{ij} = \frac{1}{\det A}(DA)_{ij} = 0$ for all $i \neq j$, too, and all the off-diagonal elements of $BA$ are zero.

**5.** By Theorem 6 generalized to a product of $p$ matrices, we have

$$\det A_1 A_2 \ldots A_p = (\det A_1)(\det A_2)\ldots(\det A_p)$$

If the product $A_1 \ldots A_p$ is invertible, then its determinant is nonzero, so none of the determinants on the right-hand side can be zero. Therefore, all the individual matrices $A_k$ are invertible, too.

Conversely, if all of the individual matrices $A_k$ are invertible, their determinants are all nonzero, so the determinant of the product $A_1 \ldots A_p$ is nonzero, too. That is, it is invertible.