Math 221 (101) Matrix Algebra October 16, 2002

Quiz #5

DUE: Wednesday, October 23 at 9:00am

Rules:

1. Quizzes are due at the beginning of class on the due date. Late quizzes will not receive credit. Remember to include your name and student number on your answers.
2. You may work with other Math 221 students and use materials such as textbooks, notes, calculators, or computers. However, you must write up your own original, complete solutions. Copying all or part of an answer is not allowed.
3. Show all work. Explain your solutions clearly. You may write in pencil if you'd like.
4. The maximum possible score is 20.

Note: Problem #2 corrected on October 17.

Solve the following five problems,[1] showing all work. Don't forget to put your name and student number on your answers.

1. Consider the following 2×2 matrices:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

(a) Calculate the determinants of the matrices. (Observe that the determinants of A and B are positive while the determinants of C and D are negative.)

(b) Consider the points of the unit square: $\mathbf{v}_1 = (0,0)$, $\mathbf{v}_2 = (1,0)$, $\mathbf{v}_3 = (1,1)$, and $\mathbf{v}_4 = (0,1)$. Plot these points, and label them "1" through "4". For each transformation—$\mathbf{x} \mapsto A\mathbf{x}$, $\mathbf{x} \mapsto B\mathbf{x}$, $\mathbf{x} \mapsto C\mathbf{x}$, and $\mathbf{x} \mapsto D\mathbf{x}$—plot the images of $\mathbf{v}_1$ through $\mathbf{v}_4$ labelling the points in order, "1" through "4".

(c) How do the images differ when the square is transformed by matrices with positive determinants (like A and B) and negative determinants (like C and D)? [Hint: The answer has to do with the ordering of the points, not their exact positions.]

2. *Correction:* The entry in row 3, column 4 of the matrix A should be 0, not 2, and the entry in row 5, column 4 should be 3, not -1.

Consider the horrible, horrible matrix:

$$A = \begin{bmatrix} 1 & 2 & 0 & 6 & 1 \\ 0 & 1 & 1 & 3 & 1 \\ 1 & 1 & -2 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ -2 & -2 & 4 & 3 & -1 \end{bmatrix}$$

Calculating the determinant directly from the definition (or from Theorem 3.1) would be difficult. However, using row reduction, we can calculate the determinant *and* the inverse all at once without too much work (less than 20 row operations).

[1] Actually, there are only four problems this week, but problem #2 counts as two because it's a lot of work.

If we augment A with a copy of the 5×5 identity matrix to form the matrix $[A \,|\, I]$ and begin to reduce it to echelon form, after 6 row replacement operations,[2] we have the following:

$$[A \,|\, I] \overset{R}{\sim} \begin{bmatrix} 1 & 2 & 0 & 6 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 9 & -1 & 2 & -2 & 0 & 0 & 1 \end{bmatrix}$$

(a) Complete the row reduction to echelon form without using row scaling operations (that is, using only row replacement and, if desired, row exchange operations—but, if you use row exchange operations, keep track of how many).

(b) Using the echelon form calculated in (a) and the boxed formula on page 188, show that the matrix is invertible and has determinant 3.

(c) Now, complete the reduction to reduced echelon form (using any row operations you want, including scaling) to calculate the inverse of A. Because that was far too much work to avoid making at least one mistake, check that your inverse works by making sure either $AA^{-1} = I$ or $A^{-1}A = I$.

3. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with $m \times n$ standard matrix A. For each of the following questions: if the answer is "yes", illustrate with an example; if the answer is "no", explain why not.

(a) If $m < n$, can T be one-to-one?

(b) If $m < n$, can T be onto?

(c) If $m > n$, can T be one-to-one?

(d) If $m > n$, can T be onto?

4. Let A be an $m \times n$ matrix. Note that the matrix A^TA is a square, $n \times n$ matrix. Assume that A^TA is invertible, and define $C = A(A^TA)^{-1}A^T$.

(a) What is the size of matrix C?

(b) Using the properties of transposes and inverses, show that $C^T = C$ and $C^2 = C$ (where C^2 is just another way of writing the product CC).

(c) Show that if $\mathbf{v}$ is any vector in the span of the columns of A, then $C\mathbf{v} = \mathbf{v}$. [Hint: We can always write such a vector $\mathbf{v}$ as $A\mathbf{x}$ for an appropriate vector $\mathbf{x}$ of weights.]

(d) Prove that for *any* vector $\mathbf{w}$, the product $C\mathbf{w}$ is always in the span of A. [Hint: This is way easier than it looks. All you need to do is show that $C\mathbf{w}$ can be written as the matrix A times some vector of weights, no matter how ugly that vector of weights looks.]

The matrix C is closely related to the statistical procedure of *least-squares* or *linear regression*. We'll talk about this more in the optional Section 6.5, if we find we have enough time to cover it.

[2] Namely, $R3 \to R3 - R1$, $R4 \to R4 - R1$, $R5 \to R5 + 2R1$, $R3 \to R3 + R2$, $R4 \to R4 + R2$, and $R5 \to R5 - 2R2$, but you certainly don't need to verify that!