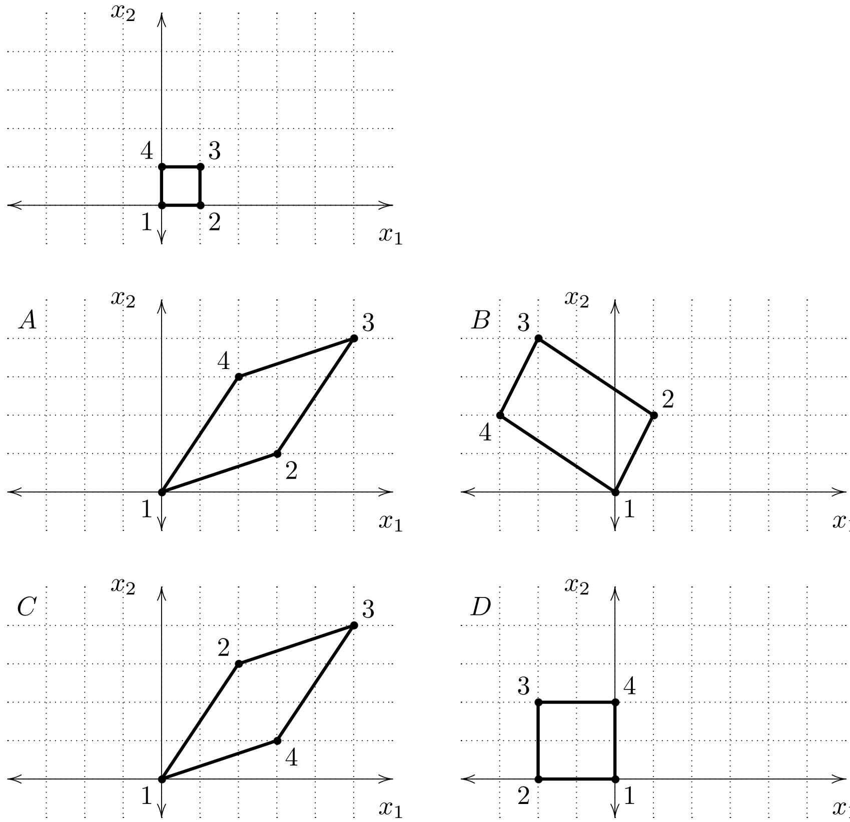
Math 221 (101) Matrix Algebra — October 25, 2002

Quiz #5 Solutions

1. (a) $|A| = \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = 3(3) - 2(1) = 7$ $\qquad |B| = \begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} = 1(2) - (-3)(2) = 8$

$|C| = \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 2(1) - 3(3) = -7$ $\qquad |D| = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -2(2) - 0(0) = -4$

(b)



(c) For the matrices with positive determinants, the transformed square has its points 1, 2, 3, and 4 in counter-clockwise order (the same order as the original square). For the matrices with negative determinants, the transformed square has its points in opposite order.

2. (a) Completing the reduction to echelon form, we get:

$$\begin{bmatrix} 1 & 2 & 0 & 6 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 9 & -1 & 2 & -2 & 0 & 0 & 1 \end{bmatrix}$$

$$\Uparrow$$

$$\xrightarrow{R5 \to R5+2R3} \begin{bmatrix} 1 & 2 & 0 & 6 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 2 & 0 & 1 \end{bmatrix}$$

$$\Uparrow$$

$$\xrightarrow{R5 \to R5+R4} \begin{bmatrix} \boxed{1} & 2 & 0 & 6 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 3 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-1} & -3 & 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-3} & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

Note that we used only row replacement operations.

(b) Since we used only row replacement operations and no row interchange (or scaling) operations, the determinant of the original matrix A is given by the product of the main diagonal of the echelon form of A (the first five columns of this augmented matrix). That is,

$$\det A = (1)(1)(-1)(-3)(1) = 3$$

Since the determinant in nonzero, the matrix is invertible.

(c) Completing the reduction to reduced echelon form, we get:

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 6 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 3 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-1} & -3 & 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-3} & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{R1\to R1-R5 \\ R2\to R2-R5 \\ R3\to R3-R5}} \begin{bmatrix} \boxed{1} & 2 & 0 & 6 & 0 & 2 & -1 & -2 & -1 & -1 \\ 0 & \boxed{1} & 1 & 3 & 0 & 1 & 0 & -2 & -1 & -1 \\ 0 & 0 & \boxed{-1} & -3 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -3 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{R1\to R1+2R4 \\ R2\to R2+R4 \\ R3\to R3-R4 \\ R4\to -\frac{1}{3}R4}} \begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & -1 \\ 0 & \boxed{1} & 1 & 0 & 0 & 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & -2 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{R3\to -R3 \\ R2\to R2-R3}} \begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -3 & -2 & -2 \\ 0 & 0 & \boxed{1} & 0 & 0 & -1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 - 2R2} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 & -2 & 1 & 4 & 5 & 3 \\ 0 & \boxed{1} & 0 & 0 & 0 & 1 & 0 & -3 & -2 & -2 \\ 0 & 0 & \boxed{1} & 0 & 0 & -1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

As expected (since we already know that A is invertible), the left half of the reduced echelon form is the identity matrix. Therefore, the right half is A^{-1}. That is,

$$A^{-1} = \begin{bmatrix} -2 & 1 & 4 & 5 & 3 \\ 1 & 0 & -3 & -2 & -2 \\ -1 & 1 & 1 & 2 & 1 \\ \frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ -1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

Double-checking the answer:

$$AA^{-1} = \begin{bmatrix} 1 & 2 & 0 & 6 & 1 \\ 0 & 1 & 1 & 3 & 1 \\ 1 & 1 & -2 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ -2 & -2 & 4 & 3 & -1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 4 & 5 & 3 \\ 1 & 0 & -3 & -2 & -2 \\ -1 & 1 & 1 & 2 & 1 \\ \frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ -1 & 1 & 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3. (a) No. The columns of A must be linearly dependent, since there are more vectors (n) than the number of elements in each vector (m). Therefore, T cannot be one-to-one.

(b) Yes. One simple example is the projection transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(\mathbf{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

As $\mathbf{x}$ varies over all vectors in $\mathbb{R}^3$, $T(\mathbf{x})$ will take on all values in $\mathbb{R}^2$.

Any linear transformation T whose standard matrix has a pivot position in every row (so its columns span all of $\mathbb{R}^m$) will work.

(c) Yes. One simple example is the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T(\mathbf{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

Obviously, no two unequal vectors can map to the same image.

Any linear transformation T whose standard matrix has linearly independent columns (that is, whose standard matrix has a pivot position in every column) will work.

(d) No. The columns of A must span $\mathbb{R}^m$. This can only happen if there is a pivot position in every row, but there can only be a maximum of n pivot positions (one in each column) and this is less than the number of rows m.

4. (a) Since A^TA is $n \times n$, its inverse $(A^TA)^{-1}$ is $n \times n$, too. Thus, C, is a product of an $m \times n$, an $n \times n$, and an $n \times m$ matrix. Therefore, it is an $m \times m$ matrix.

(b) From the properties of inverses and transposes:

$$\begin{aligned} C^T &= \big(A(A^TA)^{-1}A^T\big)^T \\ &= (A^T)^T\big((A^TA)^{-1}\big)^T(A)^T \\ &= A\big((A^TA)^T\big)^{-1}A^T \\ &= A\big((A)^T(A^T)^T\big)^{-1}A^T \\ &= A(A^TA)^{-1}A^T \\ &= C \end{aligned}$$

Note that you *can't* use $(A^TA)^{-1} = A^{-1}(A^T)^{-1}$. The matrix A may not even be square, so A^{-1} and $(A^T)^{-1}$ may not even be defined.
Similarly for the second part:

$$\begin{aligned} C^2 &= \big(A(A^TA)^{-1}A^T\big)\big(A(A^TA)^{-1}A^T\big) \\ &= A\big((A^TA)^{-1}A^TA\big)(A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}A^T \\ &= C \end{aligned}$$

(c) Since $\mathbf{v}$ is in the span of the columns of A, we can write it as $\mathbf{v} = A\mathbf{x}$ for some vector $\mathbf{x}$ of weights. Then

$$\begin{aligned} C\mathbf{v} &= CA\mathbf{x} \\ &= \big(A(A^TA)^{-1}A^T\big)A\mathbf{x} \\ &= A\big((A^TA)^{-1}A^TA\big)\mathbf{x} \\ &= A\mathbf{x} \\ &= \mathbf{v} \end{aligned}$$

(d) For any vector $\mathbf{w} \in \mathbb{R}^m$, we have

$$C\mathbf{w} = \big(A(A^TA)^{-1}A^T\big)\mathbf{w} = A\big((A^TA)^{-1}A^T\mathbf{w}\big)$$

Since $(A^TA)^{-1}A^T\mathbf{w}$ is an $n \times 1$ vector of weights (even though it's an ugly one), this shows that $C\mathbf{w}$ is some linear combination of the columns of A.