Math 221 (101) Matrix Algebra October 9, 2002

Quiz #4

DUE: Wednesday, October 16 at 9:00am

Rules:

1. Quizzes are due at the beginning of class on the due date. Late quizzes will not receive credit. Remember to include your name and student number on your answers.
2. You may work with other Math 221 students and use materials such as textbooks, notes, calculators, or computers. However, you must write up your own original, complete solutions. Copying all or part of an answer is not allowed.
3. Show all work. Explain your solutions clearly. You may write in pencil if you'd like.
4. The maximum possible score is 20.

Solve the following five problems, showing all work. Don't forget to put your name and student number on your answers.

1. Consider the matrices

$$A = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$$

and assume $k \neq 0$.

(a) Calculate the determinants of these matrices.

(b) Which of these matrices are invertible? For those that are, find their inverses and calculate the determinants of the inverses.

(c) Calculate AB and $\det(AB)$.

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 5 & 0 & k \end{bmatrix}$$

(a) Calculate the determinant of A using a cofactor expansion across a convenient row or column.

(b) For what values of k does A have an inverse?

For parts (c) and (d), assume $k = 2$.

(c) Calculate A^{-1}. Check your answer by making sure that at least one of AA^{-1} and $A^{-1}A$ is equal to the identity matrix.

(d) Use your answer in part (c) to calculate the unique solution of $A\mathbf{x} = \mathbf{b}$ for general $\mathbf{b}$ given by $\mathbf{b} = (b_1, b_2, b_3, b_4)$.

3. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates vectors clockwise $90°$ around the origin. Let A be its standard matrix. We write A^2 for the product AA, A^3 for the product AAA, and—in general—A^n for the product of n copies of A.

(a) What is the geometric interpretation of $\mathbf{x} \mapsto A^2\mathbf{x}$? of $\mathbf{x} \mapsto A^3\mathbf{x}$? of $\mathbf{x} \mapsto A^4\mathbf{x}$?

(b) Why can you conclude that $A^4 = I$, for I the 2×2 identity matrix?

(c) Is A invertible? If so, what is its inverse? Is A^2 invertible? If so, what is its inverse?

4. *[An application of linear difference equations to biology]* The E. coli bacterium is one of the most extensively studied organisms. Wild type (naturally occurring) E. coli can survive well over a wide range of temperatures, from $15°$ to $45°$C. However, heat sensitive mutants cannot survive as well at high temperatures.

For each positive integer k, let the vector

$$\mathbf{x}_k = \begin{bmatrix} w_k \\ h_k \end{bmatrix}$$

represent the number (in thousands) of wild type (w_k) and heat sensitive (h_k) bacteria in an experimental colony after k hours have elapsed.

(a) If the colony is grown at a temperature of $30°$C in the presence of a toxic chemical that causes some wild type bacteria to mutate into heat sensitive bacteria, then, each hour, every wild type bacterium produces four wild type offspring and one mutated heat sensitive offspring and every heat sensitive bacterium produces four heat sensitive offspring. Mathematically, the colony evolves according to the difference equation:

$$\mathbf{x}_{k+1} = \begin{bmatrix} w_{k+1} \\ h_{k+1} \end{bmatrix} = w_k \begin{bmatrix} 4 \\ 1 \end{bmatrix} + h_k \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Write the difference equation in matrix form $\mathbf{x}_{k+1} = A\mathbf{x}_k$ giving the matrix A explicitly. If the colony at time zero consists of 10 thousand wild type and no heat sensitive bacteria—that is, if $\mathbf{x}_0 = (10, 0)$—calculate $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$.

(b) If the colony is grown at a temperature of $40°$C in the presence of the same toxic chemical, then, each hour, every wild type bacterium still produces four wild type offspring and one mutated heat sensitive offspring. However, every heat sensitive bacterium produces only one heat sensitive offspring. Write the difference equation in matrix form $\mathbf{x}_{k+1} = A\mathbf{x}_k$. If the initial colony is $\mathbf{x}_0 = (10, 0)$, calculate $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$.

5. Let A and B be $n \times n$ matrices with A singular and B nonsingular. Prove that AB is singular. [Hint: A proof by contradiction works well here. Suppose the opposite of what we want to prove: that is, suppose that AB is nonsingular. Then, it has an inverse C. Can you use this matrix (together with the others) to construct a matrix D that satisfies $AD = I$? If so, you can apply the Invertible Matrix Theorem from Section 2.3 to get A invertible, a contradiction.]