# Quiz #4 Solutions

**1.** (a) Using the formula,

$$\det A = (-1)4 - 2(-3) = 2$$
$$\det B = 1(k) - 0(1) = k$$
$$\det C = 1(-6) - 2(-3) = 0$$

(b) $A$ and $B$ are invertible since they are the ones with non-zero determinants. Using the formula,

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$B^{-1} = \frac{1}{k}\begin{bmatrix} k & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/k & 1/k \end{bmatrix}$$

From the formula

$$\det A^{-1} = 2(-1/2) - (-1)(3/2) = 1/2$$
$$\det B^{-1} = 1(1/k) - 0(-1/k) = 1/k$$

(c) We have

$$AB = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix} = \begin{bmatrix} 1 & 2k \\ 1 & 4k \end{bmatrix}$$

and, by the formula,

$$\det AB = 1(4k) - 2k(1) = 2k$$

(which matches what we'd expect from $\det AB = (\det A)(\det B)$.

**2.** (a) Many cofactor expansions are possible. The easiest are:

- across the second row, we get:

$$\det A = -0 + 1\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & k \end{vmatrix} - 0 + 0$$

The determinant of that $3 \times 3$ matrix can be calculated by expanding across the second row:

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & k \end{vmatrix} = -0 + 1\begin{vmatrix} 1 & 1 \\ 1 & k \end{vmatrix} - 0 = k - 1$$

giving a final answer:

$$\det A = -0 + 1(k-1) - 0 + 0 = k - 1$$

- across the third row (or down the third column), we get:

$$\det A = 0 - 0 + 1 \begin{vmatrix} 1 & 4 & 1 \\ 0 & 1 & 0 \\ 1 & 5 & k \end{vmatrix} - 0$$

  Again, expanding that $3 \times 3$ matrix across the second row, we calculate its determinant to be $k - 1$ giving $\det A = k - 1$.

(b) $A$ has an inverse iff $\det A \neq 0$ iff $k - 1 \neq 0$. Therefore, $A$ has an inverse for all $k \neq 1$.

(c) We augment $A$ with a copy of the $4 \times 4$ identity matrix and reduce it to reduced echelon form:

$$\begin{bmatrix} 1 & 4 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 5 & 0 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R4 \to R4 - R1} \begin{bmatrix} 1 & 4 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R4 \to R4 - R2} \begin{bmatrix} \boxed{1} & 4 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{R1 \to R1 - R4} \begin{bmatrix} \boxed{1} & 4 & 0 & 0 & 2 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & -1 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 - 4R2} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 2 & -3 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & -1 & -1 & 0 & 1 \end{bmatrix}$$

Observe that the left half of the reduced echelon form is now the identity matrix. This is as expected, since from (b), we know that $A$ has an inverse for $k = 2$. Therefore, the right half of the reduced echelon form is the inverse of $A$:

$$A^{-1} = \begin{bmatrix} 2 & -3 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

to check our answer, simply multiply:

$$\begin{bmatrix} 2 & -3 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 5 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and verify we get the identity matrix.

(d) The unique solution of $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{p} = A^{-1}\mathbf{b}$, so:

$$\mathbf{p} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & -3 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 2b_1 - 3b_2 - b_4 \\ b_2 \\ b_3 \\ -b_1 - b_2 + b_4 \end{bmatrix}$$

**3.** (a) Because of the connection between matrix multiplication and application of transformations (see p. 101), transformation by the matrix $A^2$ is the same as applying the transformation $\mathbf{x} \mapsto A\mathbf{x}$ (clockwise rotation by 90°) and then applying the same transformation *again* to the result (another clockwise rotation by 90°).

Therefore, $\mathbf{x} \mapsto A^2\mathbf{x}$ is clockwise rotation by 180°, $\mathbf{x} \mapsto A^3\mathbf{x}$ is clockwise rotation by 270°, and $\mathbf{x} \mapsto A^4\mathbf{x}$ is clockwise rotation by 360°.

Alternatively, you could figure out the standard matrix $A$ from the description of the transformation, calculate $A^2$, $A^3$, and $A^4$, and figure out what they do directly. Their explicit values are:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) $\mathbf{x} \mapsto A^4\mathbf{x}$ corresponds to clockwise rotation 360° around the origin. This has the effect of rotating every vector through a full circle back to its original position. In other words, $\mathbf{x} \mapsto A^4\mathbf{x}$ maps every vector to itself. It's the identity transformation, and its standard matrix $A^4$ must be the identity matrix. Therefore, $A^4 = I$.

Alternatively, you could calculate it directly.

(c) In (b), we showed that $A^4 = AAAA = I$. Therefore, $A(AAA) = I$ and $(AAA)A = I$. By definition, $A$ is invertible with $A^{-1} = AAA$. Similarly, since $A^2A^2 = A^4 = I$, it follows by definition that $A^2$ is invertible with inverse $A^2$.

This is all the detail I wanted for this question, but, alternatively, if you calculated the matrices explictly, you could use the formula to calculate their inverses, and you would discover that $A^{-1}$ equals the value for $A^3$ shown above and $(A^2)^{-1}$ equals $A^2$ itself.

**4.** (a) By the definition of the matrix-vector product,

$$\mathbf{x}_{k+1} = \begin{bmatrix} w_{k+1} \\ h_{k+1} \end{bmatrix} = w_k \begin{bmatrix} 4 \\ 1 \end{bmatrix} + h_k \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} w_k \\ h_k \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix} \mathbf{x}_k$$

Therefore, we have

$$\mathbf{x}_1 = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 40 \\ 10 \end{bmatrix}$$
$$\mathbf{x}_2 = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix} \mathbf{x}_1 = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 40 \\ 10 \end{bmatrix} = \begin{bmatrix} 160 \\ 80 \end{bmatrix}$$
$$\mathbf{x}_3 = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix} \mathbf{x}_2 = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 160 \\ 80 \end{bmatrix} = \begin{bmatrix} 640 \\ 480 \end{bmatrix}$$

(b) At this higher temperature, we have

$$\mathbf{x}_{k+1} == \begin{bmatrix} w_{k+1} \\ h_{k+1} \end{bmatrix} = w_k \begin{bmatrix} 4 \\ 1 \end{bmatrix} + h_k \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} w_k \\ h_k \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x}_k$$

Therefore, we have

$$\mathbf{x}_1 = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 40 \\ 10 \end{bmatrix}$$
$$\mathbf{x}_2 = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x}_1 = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 40 \\ 10 \end{bmatrix} = \begin{bmatrix} 160 \\ 50 \end{bmatrix}$$
$$\mathbf{x}_3 = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2 = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 160 \\ 50 \end{bmatrix} = \begin{bmatrix} 640 \\ 210 \end{bmatrix}$$

**5.** Let $A$ be singular and $B$ nonsingular. Suppose that $AB$ is nonsingular. Then, it has an inverse $C$ satisfying $(AB)C = I$. That is, $A(BC) = I$, so there exists a matrix $D = BC$ satisfying $AD = I$. By the invertible matrix theorem, $A$ is invertible, a contradiction. Therefore, $AB$ must be singular.