Quiz #3

DUE: Wednesday, October 2 at 9:00am

Rules:

1. Quizzes are due at the beginning of class on the due date. Late quizzes will not receive credit. Remember to include your name and student number on your answers.
2. You may work with other Math 221 students and use materials such as textbooks, notes, calculators, or computers. However, you must write up your own original, complete solutions. Copying all or part of an answer is not allowed.
3. Show all work. Explain your solutions clearly. You may write in pencil if you'd like.
4. The maximum possible score is 20.

Solve the following five problems, showing all work. Don't forget to put your name and student number on your answers.

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by

$$T\left(\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}\right) = \begin{bmatrix}2\mathbf{x}_1+\mathbf{x}_2\\ 1\end{bmatrix} + \begin{bmatrix}-\mathbf{x}_1\\ \mathbf{x}_2-1\end{bmatrix}$$

(a) Prove that T is a linear transformation.

(b) Find the standard matrix A for the linear transformation T.

(c) On a piece of graph paper, label your x_1 and x_2 axes, and draw the rectangle with corners $\mathbf{v}_1 = (0,0)$, $\mathbf{v}_2 = (0,3)$, $\mathbf{v}_3 = (2,3)$, and $\mathbf{v}_4 = (2,0)$. Preferably using a different colored pen, draw the image of the rectangle under the transformation T.

(d) Give a geometric description of what the transformation T does to vectors in $\mathbb{R}^2$.

2. Let

$$A = \begin{bmatrix}1 & 0\\ -1 & 1\\ 2 & 1\end{bmatrix} \qquad B = \begin{bmatrix}1 & 2 & 3\\ -1 & -2 & -3\end{bmatrix} \qquad C = \begin{bmatrix}1 & 2 & 3\\ 1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}$$

and let I_2 and I_3 be the 2×2 and 3×3 identity matrices respectively. For each of the following, calculate its value or explain why it is undefined:

(a) AB (b) BA (c) ABC (d) $(AB+2I_3)C - AI_2BC - I_3C$

[Hint: For (c), you can save yourself some time by using your result from (a). For (d), you can save yourself *lots* of time by simplifying the expression first.]

3. Let $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ be three vectors in $\mathbb{R}^3$. Prove that these three vectors span all of $\mathbb{R}^3$ if and only if the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly independent.[1] [Hint: Remember Theorem 4 on page 42. If the three vectors span all of $\mathbb{R}^3$, we have some important information about the pivot positions of A. This should tell you something about whether $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}$ has a nontrivial solution or not, and that's the key to deciding whether or not those vectors are linearly independent. To prove the converse—that linear independence implies they span all of $\mathbb{R}^3$—you can use basically the same argument in reverse.]

4. Let A be a 5×3 matrix. Suppose you observe that the third column of A is the sum of the first two columns.

(a) Are the columns of A linearly dependent or independent? Why?

(b) Does $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? If so, give a specific example of one. If not, explain why not.

(c) Say that, for a particular $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent. How big is its solution set?

5. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, and let T be the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.

(a) Express the solution set of the equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

(b) Consider the equation $A\mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Note that $(1, 2)$ is a solution to this equation. Using this fact together with the result from part (a), express the solution set of this equation in parametric vector form.

(c) Graph the solution sets from parts (a) and (b), and explain their geometric relationship.

(d) What is the geometric interpretation of the transformation T? [Hint: It may help to graph points such as $(-2, -1)$, $(0, 1)$, $(1, 0)$, $(-1, 2)$, and $(4, 2)$ and see how T transforms each of them by drawing little arrows from each point $\mathbf{v}$ to its image $T(\mathbf{v})$. If that still doesn't help, adding the line $x_2 = x_1$ might reveal the answer.]

(e) Is T one-to-one? Is it onto $\mathbb{R}^2$? Justify your answer.

(f) What is the image under T of the solution set calculated in part (a)? What is the image under T of the solution set calculated in part (b)? What can you say about the image under T of the solution set of $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$ such that the equation is consistent?

[1] This is a small part of the gigantic Theorem 8 in Section 2.3 which we'll be studying in a future class.