Quiz #3 Solutions

1. (a) Note that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ 1 \end{bmatrix} + \begin{bmatrix} -x_1 \\ x_2 - 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$

This is linear by defintion, since

$$T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + v_1 + u_2 + v_2 \\ u_2 + v_2 \end{bmatrix}$$

and

$$T\left(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} cu_1 + cu_2 \\ cu_2 \end{bmatrix} = c\begin{bmatrix} u_1 + u_2 \\ u_2 \end{bmatrix} = cT\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)$$

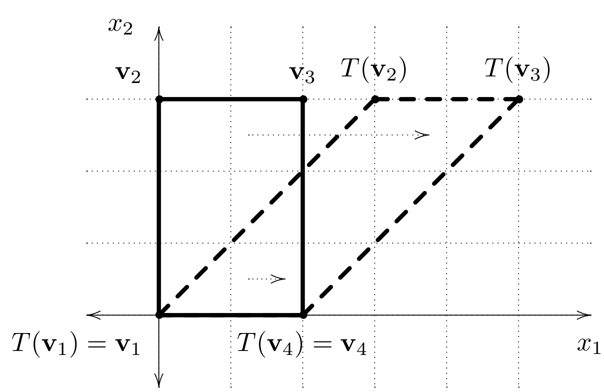
(b) Since

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

by Theorem 10, we have

$$T(\mathbf{x}) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{the standard matrix of } T}\mathbf{x}, \quad \text{for all } \mathbf{x} \in \mathbb{R}^2$$

(c)



where $T(0,0) = (0,0)$, $T(0,3) = (3,3)$, $T(2,3) = (5,3)$, and $T(2,0) = (2,0)$.

(d) T is a horizontal shear. It shift vectors horizontally an amount equal to their vertical height above the x_1 axis.

2. (a)

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \\ 1 & 2 & 3 \end{bmatrix}$$

(b)

$$BA = \begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix}$$

(c)

$$ABC = (AB)C = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 8 \\ -6 & -14 & -16 \\ 3 & 7 & 8 \end{bmatrix}$$

(d)

$$\begin{aligned} (AB + 2I_3)C - AI_2BC - I_3C &= ABC + 2I_3C - AI_2BC - I_3C \\ &= ABC + 2C - ABC - C \\ &= C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

3. If $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ span $\mathbb{R}^3$, then by Theorem 4, the matrix

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$$

has a pivot position in every row. That it, it has three pivot positions. Since it has three columns, it has a pivot position in every column. Therefore, the equation

$$A\mathbf{x} = \mathbf{0}$$

has an augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{0} \end{bmatrix}$ with pivot positions in the first three columns. Therefore, it has only the trivial solution, and $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly independent.

Conversely, if $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly independent, then $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{0} \end{bmatrix}$ has pivot positions in the first three columns. Thus, it has pivot positions in all three rows, so by Theorem 4, the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ span $\mathbb{R}^3$.

4. Write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$. We are given that $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$.

(a) They are linearly dependent by Theorem 7.

(b) Since

$$1\mathbf{a}_1 + 1\mathbf{a}_2 + (-1)\mathbf{a}_3 = \mathbf{0}$$

the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $(1, 1, -1)$.

(c) By Theorem 6, the solution set of $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \mathbf{p} + \mathbf{v}_h$$

for $\mathbf{p}$ any solution of $A\mathbf{x} = \mathbf{b}$ and all $\mathbf{v}_h$ that are solutions of $A\mathbf{x} = \mathbf{0}$.
By (b), $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions, so $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions, too.

5. (a)

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Uparrow$$

giving the general solution

$$\begin{cases} x_2 = 0 \\ x_1 \text{ free} \end{cases}$$

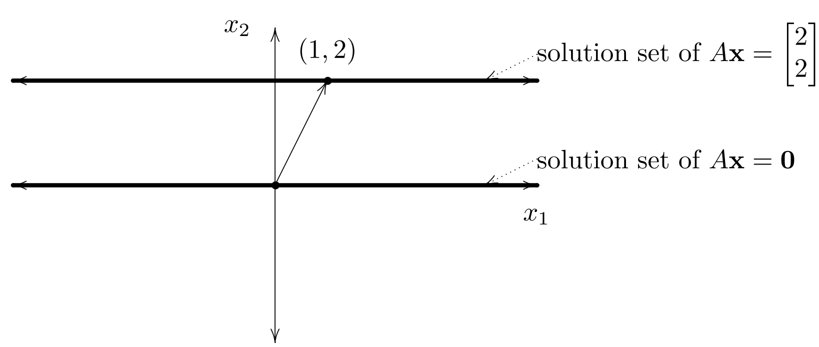
Therefore, the parametric vector form of the solution set is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_1 \text{ free}$$

(b) By Theorem 6, the solution set of $A\mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ can be written

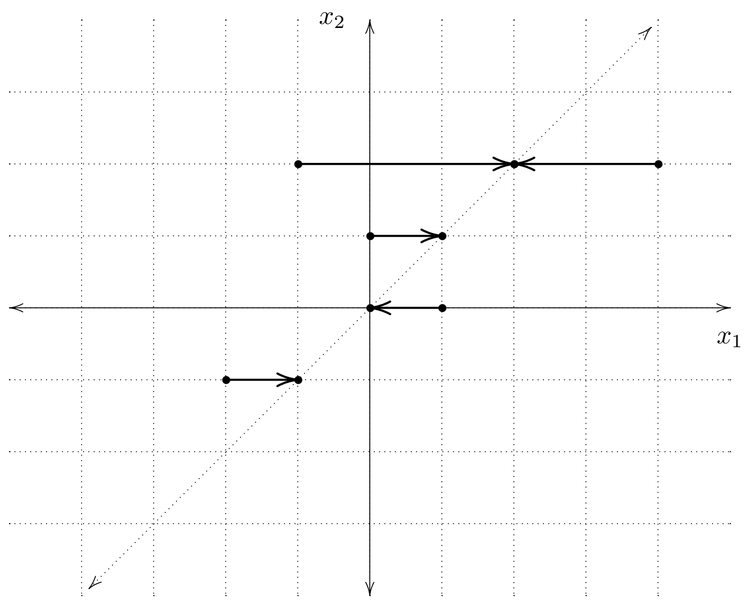
$$\mathbf{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{a specific solution}} + \underbrace{c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c_1 \in \mathbb{R}}_{\substack{\text{all solutions of the} \\ \text{homogeneous system}}},$$

(c)



The solution set in (b) is parallel to that in (a) but shifted by the vector $\mathbf{p} = (1, 2)$.

(d) Note that $T(-2, -1) = (-1, -1)$, $T(0, 1) = (1, 1)$, $T(1, 0) = (0, 0)$, $T(-1, 2) = (2, 2)$, and $T(4, 2) = (2, 2)$. Plotting these sample points:



we see that T projects each point horizontally onto the line $x_1 = x_2$.

(e) T is not one-to-one: observe that the points $(-1, 2)$ and $(4, 2)$ are both transformed to $(2, 2)$.

T is not onto $\mathbb{R}^2$: observe that all the images lie on the line $x_1 = x_2$. In particular, no vector has image $(1, 0)$ under T.

(f) If $\mathbf{x}$ is a point in the solution set of $A\mathbf{x} = \mathbf{0}$, then

$$T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$$

so all such images are equal to $\mathbf{0}$. The image of the whole solution set is the set $\{\mathbf{0}\}$.

Similarly, the image of the solution set of $A\mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is the single point $\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$.

For any $\mathbf{b}$, if $A\mathbf{x} = \mathbf{b}$ is consistent, then the image of its solution set will be $\{\mathbf{b}\}$.