Quiz #2 Solutions

1. (a) The equation is of the form $A\mathbf{x} = \mathbf{0}$, so it is homogeneous. Homogeneous systems are always consistent because they have at least one solution, the trivial solution $\mathbf{0}$.

(b) The augmented matrix is reduced as follows

$$\begin{bmatrix} 1 & -3 & -1 & 7 & 0 \\ 0 & 0 & 2 & -4 & 0 \\ 2 & -6 & -1 & 12 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R1} \begin{bmatrix} 1 & -3 & -1 & 7 & 0 \\ 0 & 0 & 2 & -4 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - \frac{1}{2}R2} \begin{bmatrix} \boxed{1} & -3 & -1 & 7 & 0 \\ 0 & 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \to \frac{1}{2}R2} \begin{bmatrix} \boxed{1} & -3 & -1 & 7 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 + R2} \begin{bmatrix} \boxed{1} & -3 & 0 & 5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives the general solution

$$\begin{cases} x_1 = 3x_2 - 5x_4 \\ x_3 = 2x_4 \\ x_2,\ x_4 \text{ free} \end{cases}$$

To get parametric vector form, we write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad x_2,\ x_4 \text{ free}$$

(c)

$$\begin{bmatrix} 1 & -3 & -1 & 7 \\ 0 & 0 & 2 & -4 \\ 2 & -6 & -1 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ -6 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 7 \\ -4 \\ 12 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$$

(d) Part (b) gives (in parametric vector form) the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Part (c) shows that $\mathbf{p} = (1, 1, 1, 1)$ is a solution of $A\mathbf{x} = (4, -2, 7)$.

By Theorem 6, the set of solutions to $A\mathbf{x} = (4, -2, 7)$ is the set of solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$ "shifted" or translated by the solution $\mathbf{p} = (1, 1, 1, 1)$.

Therefore, the solution set in parametric vector form is:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad x_2,\ x_4 \text{ free}$$

Another approach is to solve the new system from scratch, but it's a lot of extra work. In that case, you'd get the augmented matrix in reduced echelon form:

$$\begin{bmatrix} 1 & -3 & 0 & 5 & 3 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and, with a bit more work, you'd find the parametric vector form:

$$\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad x_2,\ x_4 \text{ free}$$

This doesn't have the same "extra vector" as the parametric vector form above, but in fact they generate the same set of solutions as the free variables vary over all possible values. That is, they are really the same answer.

2. (a) By Theorem 9, since $\mathbf{w}$ is the zero vector, the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. You could also use the matrix method (as used in part (b)), but that's more work.

We can express $\mathbf{w}$ as a linear combination of the others:

$$\mathbf{w} = 0\mathbf{u} + 0\mathbf{v}$$

(b) Because there are only two vectors, it's enough to note that neither is a scalar multiple of the other, so they must be linearly independent.

Alternatively, you can do it the "hard" way by checking if there's a nontrivial solution to the system with augmented matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & 0 \\ 1 & 1 & 0 \\ 8 & 4 & 0 \end{bmatrix}$$

Reducing this matrix gives

$$\begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & 0 \\ 1 & 1 & 0 \\ 8 & 4 & 0 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 8 & 4 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 - \frac{1}{4}R1} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 8 & 4 & 0 \end{bmatrix}$$

$$\xrightarrow{R4 \to R4 - 2R1} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 - R2} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \xrightarrow{R4 \to R4 - 4R2} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Because there is no free variable, this has only the trivial solution, and the set is linearly independent.

(c) The set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ contains more vectors than the dimension of the space $\mathbb{R}^2$. By Theorem 8, the set is linearly dependent.

However, to express one vector as a linear combination of the others, we really need to use the augmented matrix method anyway. Reducing the augmented matrix, we have:

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 4 & 2 & 6 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 - 4R1} \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 6 & -6 & 0 \end{bmatrix}$$

$$\xrightarrow{R2 \to \frac{1}{6}R2} \begin{bmatrix} \boxed{1} & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 + R2} \begin{bmatrix} \boxed{1} & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

This gives a general solution

$$\begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \\ x_3 \text{ free} \end{cases}$$

If we take a specific nonzero solution, like using $x_3 = 1$ to get the solution $(-2, 1, 1)$, this gives a linear dependence relation among the vectors:

$$-2\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}$$

and we can express one vector as a linear combination of the others in several different ways, including:

$$\mathbf{u}_2 = 2\mathbf{u}_1 - \mathbf{u}_3$$

3. (a) The augmented matrix has the same pivot columns as A except it might also have the rightmost column as an extra pivot column. Therefore, if the augmented matrix has more pivot columns than A, it has exactly one more pivot column than A. That it, is has 21 pivot columns.

In this case, since the rightmost pivot column is a pivot column, the matrix equation is inconsistent.

(b) If A has 50 pivot positions, it has a pivot position in every row. Thus, there is no "room" for an extra pivot position in the rightmost column of the augmented matrix. Therefore, the system is consistent. Since the first 50 columns are pivot columns, there are no free variables. Therefore, there is a unique solution, and the solution set is of size one.

4. (a) $\mathbf{v}_4 = -2\mathbf{v}_1 + \mathbf{v}_2 + 0\mathbf{v}_3$

(b) Any linear combination of the four vectors can be written

$$\begin{aligned} &c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 \\ &\quad = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4(-2\mathbf{v}_1 + \mathbf{v}_2 + 0\mathbf{v}_3) \\ &\quad = (c_1 - 2c_4)\mathbf{v}_1 + (c_2 + c_4)\mathbf{v}_2 + c_3\mathbf{v}_3 \end{aligned}$$

so it is also a linear combination of the first three vectors.

Every linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is also a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$. Therefore, every element of the set $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also in the set $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

However, because of the above result, every linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ can be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Therefore, every element of the set $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also in the set $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Therefore, these two sets contain the same elements and are equal.

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, we can't write $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ (by Theorem 7). However, linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$ can be written as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ alone because $\mathbf{v}_4 = -2\mathbf{v}_1 + \mathbf{v}_2$. Thus, $\mathbf{v}_3$ can't be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$ either.

In particular, $\mathbf{v}_3$ is a vector in $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ that isn't in $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$, so they can't be the same set.

5. (a) Row reducing, we have

$$\begin{array}{ccc} \begin{bmatrix} 1 & h & 2 \\ 3 & 4 & k \end{bmatrix} & \xrightarrow{R2 \to R2 - 3R1} & \begin{bmatrix} 1 & h & 2 \\ 0 & 4-3h & k-6 \end{bmatrix} \\ \Uparrow \quad\quad & & \Uparrow \quad\quad\quad\quad \end{array}$$

By Theorem 4, the columns of A span $\mathbb{R}^2$ iff every row contains a pivot position. The columns of A will *fail* to span $\mathbb{R}^2$ iff the second row is all zeros: if $4-3h=0$ and $k-6=0$ or, in other words, if $h=4/3$ and $k=6$.

(b) The span of the columns of A is the set of all $\mathbf{b}$ for which

$$A\mathbf{x}=\mathbf{b}$$

has a solution. Row reducing, we have

$$\begin{bmatrix} 1 & 4/3 & 2 & b_1 \\ 3 & 4 & 6 & b_2 \end{bmatrix} \xrightarrow{R2 \to R2 - 3R1} \begin{bmatrix} 1 & 4/3 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 3b_1 \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

Note that this has a solution iff $b_2-3b_1=0$. Therefore, the span of A is the set of all $\mathbf{b}$ satisfying $b_2-3b_1=0$ which we can write in vector parametric form as

$$\mathbf{b}=c\begin{bmatrix}1\\3\end{bmatrix}, \quad c\in\mathbb{R}$$

For an example of a vector *not* in the span, just take any vector that violates this condition. The vector $(0,1)$ works since its second element isn't three times its first element.