Midterm #2

Please read the instructions carefully before beginning.

Instructions:

1. **Write your name, student number, and "Math 221 Sect 101" on the front page of your answer booklet.**
2. This is a closed-book exam. No textbooks, notes, formula sheets, or calculators are permitted.
3. Write your complete answers in your answer booklet(s). Additional answer booklets and scrap paper are available on request.
4. Explain your solutions clearly, showing all your work. Correct answers with insufficient, unclear, or incorrect work will not receive full credit.
5. **At the end of the test, return this question paper with your answer booklet.** Answer booklets without a question paper will not receive credit.

Solve the following **five** problems worth a total of 100 marks, showing all work.

1. (20 marks) Let A be given by

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) (15 marks) Calculate A^{-1}.

(b) (5 marks) Use part (a) to find the unique solution to

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

2. (25 marks) Let A and B be given by

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 11 & 7 & -2 & 0 \\ 0 & -1 & 4 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 4 & 1 & -11 & 0 \\ 0 & 3 & 5 & -3 \end{bmatrix}$$

TURN OVER—THE TEST CONTINUES ON THE REVERSE!

(a) (10 marks) Find the determinants of A and B.

(b) (10 marks) Calculate each of the following:

(i) $\det(AB)$	(ii) $\det(2A)$	(iii) $\det(A^3)$
(iv) $\det(ABA^{-1})$	(v) $\det(A^T A)$	

(c) (5 marks) Let U be an echelon form of B produced by 3 row replacements, 2 row interchanges, and 1 row scaling ($R3 \to -\frac{1}{3}R3$). Calculate the determinant of U.

3. (25 marks) Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 & 1 & 4 \end{bmatrix}$$

(a) (10 marks) Find a basis for $\operatorname{Col} A$.

(b) (15 marks) Find a basis for $\operatorname{Nul} A$.

4. (20 marks) Consider the triangular matrix

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) (2 marks) What are the two distinct eigenvalues of A?

(b) (15 marks) For each eigenvalue, find a basis for the associated eigenspace.

(c) (3 marks) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

5. (10 marks) Let A be an invertible, 3×3 matrix. Prove that if $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{R}^3$ then $\{A\mathbf{b}_1, A\mathbf{b}_2, A\mathbf{b}_3\}$ is also a basis for $\mathbb{R}^3$.

[Hint: Let $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$ so that $AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix}$. You may use (without proof) the fact that if A and B are invertible, then AB is invertible.]

Reminder: Return this question paper with your answer booklet.