Midterm #2 Solutions

1. (a) Augmenting with the identity matrix and reducing:

$$\begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R3 \to R3 - R1} \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - R2} \begin{bmatrix} \boxed{1} & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 - 2R3} \begin{bmatrix} \boxed{1} & 1 & 0 & 0 & 3 & 2 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 - R2} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 3 & 1 & -2 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 \end{bmatrix}$$

This gives the answer

$$A^{-1} = \begin{bmatrix} 3 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) The unique solution is given by

$$\mathbf{p} = A^{-1}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}3 & 1 & -2 & 0\\ 0 & 1 & 0 & 0\\ -1 & -1 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

2. (a) Since A is a triangular matrix, its determinant is the product of the elements along the main diagonal

$$\det A = (-1)(2)(-2)(-1) = -4$$

The determinant of B can be calculated by a cofactor expansion down the first column

$$\det B = 0 - 0 + 4\begin{vmatrix}1 & 2 & -1\\ 1 & -1 & 2\\ 3 & 5 & -3\end{vmatrix} - 0 \tag{1}$$

The determinant of the submatrix can be calculated by a cofactor expansion or by the row reduction method. Both take about the same amount of work. A cofactor expansion across the first row gives

$$\begin{aligned}\begin{vmatrix}1 & 2 & -1\\ 1 & -1 & 2\\ 3 & 5 & -3\end{vmatrix} &= 1\begin{vmatrix}-1 & 2\\ 5 & -3\end{vmatrix} - 2\begin{vmatrix}1 & 2\\ 3 & -3\end{vmatrix} + (-1)\begin{vmatrix}1 & -1\\ 3 & 5\end{vmatrix}\\ &= (3-10) - 2(-3-6) - (5-(-3)) = -7 + 18 - 8 = 3\end{aligned}$$

Alternatively, the row reduction method gives

$$\begin{bmatrix}1 & 2 & -1\\ 1 & -1 & 2\\ 3 & 5 & -3\end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix}1 & 2 & -1\\ 0 & -3 & 3\\ 3 & 5 & -3\end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - 3R1} \begin{bmatrix}1 & 2 & -1\\ 0 & -3 & 3\\ 0 & -1 & 0\end{bmatrix} \xrightarrow{R3 \to R3 - \frac{1}{3}R2} \begin{bmatrix}\boxed{1} & 2 & -1\\ 0 & \boxed{-3} & 3\\ 0 & 0 & -1\end{bmatrix}$$

The determinant of the echelon form is $(1)(-3)(-1) = 3$, and since the reduction involved only row replacement operations (which don't change the determinant), this gives the same answer 3.

In any event, once the determinant of the submatrix is known, the determinant of the original matrix B may be calculated from (1) as

$$\det B = 4(3) = 12$$

(b) i. $\det(AB) = (\det A)(\det B) = (-4)(12) = -48$
ii. $\det(2A) = 2^4 \det A = 16(-4) = -64$
iii. $\det(A^3) = (\det A)^3 = (-4)^3 = -64$
iv. $\det(ABA^{-1}) = (\det A)(\det B)\left(\frac{1}{\det A}\right) = \det B = 12$
v. $\det(A^T A) = (\det A^T)(\det A) = (\det A)(\det A) = (-4)(-4) = 16$

(c) The transformation from B to U involves 3 row replacements (which don't affect the determinant), 2 row interchanges (which change the sign of the determinant twice, resulting in no net change), and a row scaling by $-\frac{1}{3}$ which scales the determinant by the same amount. Therefore, the determinant of B is, in all, simply scaled by $-\frac{1}{3}$, giving:

$$\det U = -\frac{1}{3}\det B = -\frac{1}{3}(12) = -4$$

3. (a) We must find the pivot columns of A. Reducing to echelon form,

$$\begin{bmatrix} 0 & 1 & -1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R4} \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 4 \\ 0 & 1 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R4 \to R4 - R2} \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 4 \\ 0 & 1 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{R4 \to R4 + R3} \begin{bmatrix} \boxed{1} & 2 & -1 & 0 & 1 & 4 \\ 0 & \boxed{1} & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon form shows the pivot positions of the original matrix are in its first, second, and fifth columns:

$$A = \begin{bmatrix} \boxed{0} & 1 & -1 & 2 & 0 & 1 \\ 0 & \boxed{1} & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 1 & 2 & -1 & 0 & 1 & 4 \end{bmatrix}$$

so these columns form a basis for $\operatorname{Col} A$:

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(b) To find a basis for Nul A, we augment the matrix with a zero vector, and continue the reduction to reduced echelon form:

$$\begin{bmatrix} \boxed{1} & 2 & -1 & 0 & 1 & 4 & 0 \\ 0 & \boxed{1} & -1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - R3} \begin{bmatrix} \boxed{1} & 2 & -1 & 0 & 0 & 3 & 0 \\ 0 & \boxed{1} & -1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R2 \to R2 - R3} \begin{bmatrix} \boxed{1} & 2 & -1 & 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - 2R2} \begin{bmatrix} 1 & 0 & 1 & -4 & 0 & 1 & 0 \\ 0 & \boxed{1} & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives a general solution of

$$\begin{cases} x_1 = -x_3 + 4x_4 - x_6 \\ x_2 = x_3 - 2x_4 - x_6 \\ x_5 = -x_6 \\ x_3,\ x_4,\ x_6 \text{ free} \end{cases}$$

and a parametric vector form of

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -x_3 + 4x_4 - x_6 \\ x_3 - 2x_4 - x_6 \\ x_3 \\ x_4 \\ -x_6 \\ x_6 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad x_3,\ x_4,\ x_6 \text{ free}$$

giving a basis for Nul A of

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

4. (a) Because this is a triangular matrix, its eigenvalues are the elements along the main diagonal. This gives two distinct eigenvalues 2 and 1.

(b) **For** $\lambda = 2$**,** we calculate a basis for Nul$(A - 2I)$. The coefficient matrix is

$$A - 2I = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and we reduce the augmented matrix as follows

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{R2 \to R2 + \frac{1}{2}R1} \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R2} \begin{bmatrix} 0 & \boxed{2} & -1 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R2 \to -2R2} \begin{bmatrix} 0 & \boxed{2} & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 + R2} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to \frac{1}{2}R1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

This gives general solution

$$\begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_1 \text{ free} \end{cases}$$

with corresponding vector parametric form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_1 \text{ free}$$

giving a basis for the eigenspace (associated with $\lambda = 2$) of

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

For $\lambda = 1$**,** we calculate a basis for Nul$(A - I)$. The coefficient matrix is

$$A - 2I = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the augmented matrix is already in reduced echelon form

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives general solution

$$\begin{cases} x_1 = -2x_2 + x_3 \\ x_2,\ x_3 \text{ free} \end{cases}$$

with corresponding vector parametric form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_2,\ x_3 \text{ free}$$

giving a basis for the eigenspace (associated with $\lambda = 1$) of

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) By the Diagonalization Theorem, we put the eigenvectors from the calculated bases into P:

$$P = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the eigenvalues in the same order into D:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Alternatively, we could build P in a different order, as long as D also reflected the change. For example,

$$P = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

will work, too.

5. Since $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{R}^3$, the vectors are linearly independent. Therefore, the square matrix

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$$

is invertible by the Invertible Matrix Theorem. Since A is invertible by assumption, it follows from the result given in the hint that

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix}$$

is invertible.

However, by the Invertible Matrix Theorem, this implies that the columns of AB span $\mathbb{R}^3$ and are linearly independent. Therefore,

$$\{A\mathbf{b}_1, A\mathbf{b}_2, A\mathbf{b}_3\}$$

is a basis for $\mathbb{R}^3$ by definition.