Midterm #1

Please read the instructions carefully before beginning.

Instructions:

1. **Write your name, student number, and "Math 221 Sect 101" on the front page of your answer booklet.**
2. This is a closed-book exam. No textbooks, notes, "cheat sheets," or calculators are permitted.
3. Write your complete answers in your answer booklet(s). Additional answer booklets and scrap paper are available on request.
4. Explain your solutions clearly, showing all your work. Correct answers with insufficient, unclear, or incorrect work will not receive full credit.

Solve the following **five** problems worth a total of 100 marks, showing all work.

1. (25 marks) Let

$$A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 2 & 1 & 0 & 3 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

(a) (10 marks) Find the solution set of the homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$. Express your answer in parametric vector form. Give a specific example of a nontrivial solution.

(b) (10 marks) Find all values of k (if any) for which $(1, 2, k)$ is in the span of the columns of A. Do the columns of A span $\mathbb{R}^3$? Why or why not?

(c) (5 marks) You see (you may assume) that:

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$$

Without doing additional calculation, write down the solution to the nonhomogeneous equation

$$A\mathbf{x} = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$$

in parametric vector form.

2. (20 marks) Consider the following vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

TURN OVER—THE TEST CONTINUES ON THE REVERSE!

(a) (10 marks) Show that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Explain your reasoning clearly.

(b) (5 marks) Is there a vector $\mathbf{d}$ in $\mathbb{R}^3$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{d}\}$ is linearly *dependent*? Justify your answer.

(c) (5 marks) Is there a vector $\mathbf{u}$ in $\mathbb{R}^3$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{u}\}$ is linearly *independent*? Justify your answer.

3. (10 marks) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation whose standard matrix is given by

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) (5 marks) Is T onto $\mathbb{R}^3$? Explain your answer.

(b) (5 marks) Is T one-to-one? Explain your answer.

4. (30 marks) Suppose the 5×6 augmented matrix of a homogeneous matrix equation has an echelon form

$$\begin{bmatrix} 0 & 17 & 8 & 13 & 0 & 0 \\ 0 & 0 & 71 & 18 & -80 & 0 \\ 0 & 0 & 0 & -5 & 11 & 0 \\ 0 & 0 & 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (5 marks) How many solutions does the system have? Explain.

(b) (10 marks) Are the columns of the 5×5 *coefficient* matrix (that is, the original coefficient matrix, not the echelon form shown above) linearly dependent or independent? Explain your answer.

(c) (10 marks) Can the solution set of the matrix equation be written as the span of a single vector? Why or why not?

(d) (5 marks) Let the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, and $\mathbf{e}_5$ be the vectors in $\mathbb{R}^5$ such that each $\mathbf{e}_i$ has a one in the ith entry and zeros elsewhere. Which of these vectors, if any, is in the solution set? Justify your answer.

5. (15 marks) Let A be an $m \times n$ matrix whose columns are linearly independent, and let B be an $n \times p$ matrix whose columns are also linearly independent.

(a) (3 marks) Prove that if $A\mathbf{y} = \mathbf{0}$ then $\mathbf{y} = \mathbf{0}$. The same result will hold for B (if $B\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$).

(b) (12 marks) Using (a), or by some other means, prove that the columns of the matrix product AB are linearly independent.