# Midterm #1 Solutions

**1.** (a) The augmented matrix for this equation is reduced as follows:

$$\begin{bmatrix} 2 & 0 & 0 & 2 & 0 \\ 2 & 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 2 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 2 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - 2R2} \begin{bmatrix} 2 & 0 & 0 & 2 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to \frac{1}{2}R1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution may be written:

$$\begin{cases} x_1 = -x_4 \\ x_2 = -x_4 \\ x_3,\ x_4 \text{ free} \end{cases}$$

In parametric vector form, it becomes:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_4 \\ -x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3,\ x_4 \text{ free}$$

To give an example of a specific nontrivial solution, substitute any values for $x_3$ and $x_4$ that aren't both zero. For example, $x_3 = 1$, $x_4 = 0$ gives:

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(b) To check if a vector is in the span of the columns of $A$, we make an augmented matrix by augmenting $A$ with that vector and reducing it:

$$\begin{bmatrix} 2 & 0 & 0 & 2 & 1 \\ 2 & 1 & 0 & 3 & 2 \\ 0 & 2 & 0 & 2 & k \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 2 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & k \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - 2R2} \begin{bmatrix} 2 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & k-2 \end{bmatrix}$$

This is consistent iff $k = 2$. Therefore, $k = 2$ is the only value for which $(1, 2, k)$ is in the span.

The columns of $A$ do not span $\mathbb{R}^3$ because, as we've just shown, not all vectors in $\mathbb{R}^3$ are in the span.

Alternatively, you could look at the pivot positions of the *coefficient* matrix (not the augmented matrix). The coefficient matrix $A$ has pivot positions in the first and second rows but not the third, so by a theorem (Theorem 4), the columns of $A$ do not span $\mathbb{R}^3$.

(c) The easy way to solve this is to note that $\mathbf{p} = (1, 1, 1, 1)$ is a solution to this nonhomogeneous equation. Then, a theorem (Theorem 6) applies, and the solution set of the nonhomogeneous equation is just the solution set of the homogeneous equation (from part (a)) shifted by $\mathbf{p}$. That is,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3,\ x_4 \text{ free}$$

**2.** (a) To check if a set of vectors is linearly dependent or independent, we make an augmented matrix with those vectors as its left columns and a rightmost columns of all zeros. The resulting matrix is reduced as follows:

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 4 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 - R1} \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 + \frac{1}{2}R2} \begin{bmatrix} \boxed{1} & 3 & 2 & 0 \\ 0 & \boxed{4} & 2 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{bmatrix}$$

Because there are no free variables, the trivial solution is the only solution, so the vectors are linearly independent.

(b) Yes. We can take $\mathbf{d}$ to be any linear combination of the other vectors. Even $\mathbf{d} = \mathbf{0}$ will work.

(c) No. Four vectors in $\mathbb{R}^3$ can't be linearly independent by a theorem (Theorem 8).

**3.** (a) Because $A$ has a pivot position in every row, by a theorem (Theorem 4), the columns of $A$ span $\mathbb{R}^3$, so by another theorem (Theorem 12) $T$ is onto $\mathbb{R}^3$.

(b) Because the augmented matrix $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ represents a system with no free variables (pivot positions in all columns but the rightmost one), the columns of $A$ are linearly independent. By a theorem (Theorem 12), $T$ is one-to-one.

**4.** (a) Because the rightmost column has no pivot position, the system is consistent. Because the leftmost column has no pivot position, the variable $x_1$ is free. Therefore, the system has an infinite number of solutions.

(b) The echelon matrix given is the echelon form of the augmented matrix associated with the system

$$x_1\mathbf{a}_1 + \cdots + x_5\mathbf{a}_5 = 0$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_5$ are the columns of the original coefficient matrix. Because there are an infinite number of solutions, there is at least one nontrivial solution. Therefore, by definition, the columns of the coefficient matrix are linearly dependent.

(c) The text *does* mention (on page 50) that a homogeneous system with one free variable has a solution set that's a line through the origin (and so the span of a single vector).

If you didn't see that, it is easiest to answer this question by writing down the reduced echelon form of the augmented matrix. Actually *doing* the row reduction is difficult (though it is possible to do in just a few minutes, even without a calculator—in fact, a calculator would just get in the way).

However, there's a shortcut. The reduced echelon form will have pivot positions in the same positions as the echelon form. However, those pivot positions will contain 1s and all entries above and below the pivot positions will be 0. The left and rightmost columns will remain all zeros. Therefore, the only possible form for the reduced echelon matrix is:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the general solution

$$\begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \\ x_5 = 0 \\ x_1 \text{ free} \end{cases}$$

Written in parametric vector form, it becomes

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, *yes*, the solution set of this system can be written as the span of a single vector, namely $(1, 0, 0, 0, 0)$.

(d) Part (c) states that the solution set is the span of $\mathbf{e}_1 = (1, 0, 0, 0, 0)$. Out of all the vectors given, only $\mathbf{e}_1$ itself is in this span, so only $\mathbf{e}_1$ is in the solution set.

**5.** (a) If the columns of $A$ are linearly independent, then $A\mathbf{y} = \mathbf{0}$ has only the trivial solution. Therefore, if $A\mathbf{y} = \mathbf{0}$ for some vector $\mathbf{y}$, it must be true that $\mathbf{y} = \mathbf{0}$.

(b) Several different proofs are possible. The one I had in mind is this:

It is enough to prove that $(AB)\mathbf{x} = \mathbf{0}$ has only the trivial solution. Now, if $(AB)\mathbf{x} = \mathbf{0}$ for some vector $\mathbf{x}$, then $A(B\mathbf{x}) = \mathbf{0}$. By part (a), if $A$ times a vector is zero, that vector must be zero, so we have $B\mathbf{x} = \mathbf{0}$. But if $B\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$ for the same reason. Therefore, $(AB)\mathbf{x} = \mathbf{0}$ is only true for the trivial solution $\mathbf{x} = \mathbf{0}$, and the columns of $AB$ must be linearly independent.