Review Session #2 Diagonalization Example

Suppose we want to diagonalize

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

The first step is to find the eigenvalues. Unless the matrix is triangular (so we can read the eigenvalues off the main diagonal), we need to use the characteristic polynomial. The characteristic polynomial is calculated as follows:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 3 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{bmatrix}\right)$$

To calculate this determinant, we'll do a cofactor expansion across the second row:

$$\begin{vmatrix} 1-\lambda & 3 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{vmatrix} = -0 + (2-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 0 = (2-\lambda)(1-\lambda)(1-\lambda)$$

which is, fortunately, already factored. Therefore, the distinct eigenvalues are 2 and 1.

Now, we need to find a basis for *each* eigenspace.

- **For** $\lambda = 1$**,** we need to find a basis for the null space of the matrix

$$A - 1I = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Augmenting and reducing:

$$\begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \xrightarrow[R3 \to R3 - 2R1]{R2 \to R2 - 3R1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives the general solution

$$\begin{cases} x_2 = 0 \\ x_1,\, x_3 \text{ free} \end{cases}$$

and parametric vector form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad x_1,\, x_3 \text{ free}$$

Therefore, a basis for this null space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- **For $\lambda = 2$,** we need to find a basis for the null space of the matrix

$$A - 2I = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

Augmenting and reducing:

$$\begin{bmatrix} -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} -1 & 3 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \to \frac{1}{2}R2} \begin{bmatrix} -1 & 3 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 - 3R2} \begin{bmatrix} -1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to -R1} \begin{bmatrix} 1 & 0 & -3/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives the general solution

$$\begin{cases} x_1 = \frac{3}{2}x_3 \\ x_2 = \frac{1}{2}x_3 \\ x_3 \text{ free} \end{cases}$$

and parametric vector form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x_3 \\ \frac{1}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \qquad x_3 \text{ free}$$

Therefore, a basis for this null space is

$$\left\{ \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix} \right\}$$

Now, combining all our bases, we have a (linearly independent) set of three eigenvectors:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix} \right\}$$

Because A is 3×3, these are enough vectors for A to be diagonalizable.

We build the invertible matrix P using these vectors, in any order, for its columns:

$$P = \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 0 & 1/2 \\ 0 & 1 & 1 \end{bmatrix}$$

and we construct the diagonal matrix D using the eigenvalues in the order corresponding to the eigenvector order used for P:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$