Review Session #1 Examples

1. Suppose we need to write the parametric vector form of the solution set of

$$\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

First, the augmented matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}$$

Notice that we have a whole lot of zeros in there, including a leftmost column of all zeros. However, our row reduction algorithm works much as it always has. The first step of the algorithm is find the leftmost *nonzero* column, and make its top element a pivot position.

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}$$
$$\Uparrow$$

Now, since that pivot position contains a zero, we need to interchange rows to get a nonzero entry. Then, we zero out all rows below that pivot.

$$\xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 - \frac{1}{2}R1} \begin{bmatrix} 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

Now, we cover the top row, and again we must find the leftmost nonzero column (ignoring the top row) and make the top entry the pivot:

$$\begin{bmatrix} 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\Uparrow$$

This one is already nonzero and it has no nonzero entries below it, so we're done. This matrix is in echelon form. We reduce it further to reduced echelon form in three steps:

$$\begin{bmatrix} 0 & \boxed{2} & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \to \frac{1}{2}R2} \begin{bmatrix} 0 & \boxed{2} & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 - 4R2} \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to \frac{1}{2}R1} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The final reduced echelon matrix with the pivot positions boxed is:

$$\begin{bmatrix} 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot positions correspond to the basic variables x_2 (second column) and x_4 (fourth column). All other variables (x_1 and x_3) are free. The general solution is therefore:

$$\begin{cases} x_2 = 0 \\ x_4 = 0 \\ x_1,\ x_3 \text{ free} \end{cases}$$

Note that x_1 and x_3 *are* free variables, even if they don't appear in the right-hand sides of any of the equations in the general solution. Also, this system has an infinite number of solutions, because even though x_2 and x_4 will always be zero, we can pick x_1 and x_3 anyway we want, generating a different solution every time. For example, $(1, 0, 2, 0)$ is a solution. So is $(50, 0, -5, 0)$.

To get parametric form, we write out the vector $\mathbf{x}$ and circle the variables that are basic.

Then, we substitute those in from the general solution:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix}$$

Finally, we gather the x_1 and x_3 terms together and finish the job:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_1,\, x_3 \text{ free}$$

Don't forget to mention that x_1 and x_3 are free!

2. Suppose we want to know for what values of h and k the vector $\begin{bmatrix} 0 \\ h \\ k \end{bmatrix}$ is in the span of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. As with all problems of this type, we construct an augmented matrix with the spanning vectors as the left columns and the vector that may or may not be in the span as the rightmost column:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & h \\ 1 & 1 & k \end{bmatrix}$$

We only need to reduce this to echelon form to find out if it's consistent:

$$\begin{array}{c} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & h \\ 1 & 1 & k \end{bmatrix} \\ \Uparrow \quad\quad\quad\quad \end{array} \xrightarrow{R3 \to R3 - R1} \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & h \\ 0 & 2 & k \end{bmatrix} \\ \Uparrow \quad\quad\quad\quad \end{array}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & h \\ 0 & 2 & k \end{bmatrix} \xrightarrow{R3 \to R3 - R2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & h \\ 0 & 0 & h-k \end{bmatrix}$$

This is consistent iff $h - k = 0$ iff $h = k$. Therefore, $(0, h, k)$ is in the span of those two other vectors iff $h = k$. So, $(0, 1, 1)$ and $(0, 2, 2)$ are in the span, but $(0, 1, 2)$ isn't.

3. Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a transformation with standard matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and we want to know whether T is onto, one-to-one, neither, or both.

We begin by reducing the matrix to echelon form to find its pivot positions.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

To find out if T is onto $\mathbb{R}^2$, all we need to decide is whether or not there's a pivot position in every row of A. There *is* a pivot position in every row, so T is onto.

To find out if T is one-to-one, we need to determine if the augmented matrix $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ has one or an infinite number of solutions. In echelon form, that augmented matrix will be the echelon matrix above with the zero column tacked on the right:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Notice that there's no pivot position in the third column, so that's a free variable. Therefore, there are infinitely many solutions (so the columns of A are linearly dependent) so T is *not* one-to-one.

Therefore, T is onto but not one-to-one.