## Homework Set \#12 Solutions

## Exercises 6.5 (p. 411)

Assignment: Do \#17, 18, 3, 5, 7
17. (a) True. (p. 404)
(b) True. (p. 405)
(c) False. It should read $\|\mathbf{b}-A \mathbf{x}\| \geq\|\mathbf{b}-A \hat{\mathbf{x}}\|$.
(d) True. (Theorem 6.13, p. 406)
(e) True. (Theorem 6.14, p. 408)
18. (a) True. (p. 404)
(b) False. Instead, it is the vector $\mathbf{x}$ such that $A \mathbf{x}$ is the point in the column space closest to $\mathbf{b}$.
(c) True. This just says that $A \mathbf{x}$ is the orthogonal projection of $\mathbf{b}$ onto $\operatorname{Col} A$.
(d) False. This is true only if $A^{T} A$ is invertible.
(e) False, in the sense that some "ill-conditioned" matrices will cause problems. See page 409.
(f) True, perhaps, if you're a computer. False, if you're human. Humans are better off using the normal equations.
3. We have

$$
A^{T} A=\left[\begin{array}{rrrr}
1 & -1 & 0 & 2 \\
-2 & 2 & 3 & 5
\end{array}\right]\left[\begin{array}{rr}
1 & -2 \\
-1 & 2 \\
0 & 3 \\
2 & 5
\end{array}\right]=\left[\begin{array}{rr}
6 & 6 \\
6 & 42
\end{array}\right] \quad A^{T} \mathbf{b}=\left[\begin{array}{rrrr}
1 & -1 & 0 & 2 \\
-2 & 2 & 3 & 5
\end{array}\right]\left[\begin{array}{r}
3 \\
1 \\
-4 \\
2
\end{array}\right]=\left[\begin{array}{r}
6 \\
-6
\end{array}\right]
$$

so we must solve the system of normal equations

$$
\begin{aligned}
A^{T} A \hat{\mathbf{x}} & =A^{T} \mathbf{b} \\
{\left[\begin{array}{rr}
6 & 6 \\
6 & 42
\end{array}\right] \hat{\mathbf{x}} } & =\left[\begin{array}{r}
6 \\
-6
\end{array}\right]
\end{aligned}
$$

A few row operations give

$$
\left[\begin{array}{rrr}
6 & 6 & 6 \\
6 & 42 & -6
\end{array}\right] \xrightarrow{\begin{array}{l}
R 1 \rightarrow \frac{1}{6} R 1 \\
R 2 \rightarrow \frac{1}{6} R 2
\end{array}\left[\begin{array}{rrr}
1 & 1 & 1 \\
1 & 7 & -1
\end{array}\right] \xrightarrow{R 2 \rightarrow R 2-R 1}\left[\begin{array}{rrr}
1 & 1 & 1 \\
0 & 6 & -2
\end{array}\right]}
$$

$$
\xrightarrow{R 2 \rightarrow \frac{1}{6} R 2}\left[\begin{array}{rrr}
1 & 1 & 1 \\
0 & 1 & -1 / 3
\end{array}\right] \xrightarrow{R 1 \rightarrow R 1-R 2}\left[\begin{array}{rrr}
1 & 0 & 4 / 3 \\
0 & 1 & -1 / 3
\end{array}\right]
$$

giving the unique least-squares solution

$$
\hat{\mathbf{x}}=\left[\begin{array}{r}
4 / 3 \\
-1 / 3
\end{array}\right]
$$

5. We have

$$
A^{T} A=\left[\begin{array}{llll}
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{array}\right]\left[\begin{array}{lll}
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 1
\end{array}\right]=\left[\begin{array}{lll}
4 & 2 & 2 \\
2 & 2 & 0 \\
2 & 0 & 2
\end{array}\right] \quad A^{T} \mathbf{b}=\left[\begin{array}{llll}
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{array}\right]\left[\begin{array}{l}
1 \\
3 \\
8 \\
2
\end{array}\right]=\left[\begin{array}{r}
14 \\
4 \\
10
\end{array}\right]
$$

giving the normal equations

$$
\left[\begin{array}{lll}
4 & 2 & 2 \\
2 & 2 & 0 \\
2 & 0 & 2
\end{array}\right] \hat{\mathbf{x}}=\left[\begin{array}{r}
14 \\
4 \\
10
\end{array}\right]
$$

which have solution

$$
\begin{array}{cc}
{\left[\begin{array}{rrrr}
4 & 2 & 2 & 14 \\
2 & 2 & 0 & 4 \\
2 & 0 & 2 & 10
\end{array}\right] \xrightarrow{R 1 \rightarrow \frac{1}{2} R 1}\left[\begin{array}{rrrr}
2 & 1 & 1 & 7 \\
2 & 2 & 0 & 4 \\
2 & 0 & 2 & 10
\end{array}\right] \xrightarrow{\substack{R 2 \rightarrow R 2-R 1 \\
R 3 \rightarrow R 3-R 1}}\left[\begin{array}{rrrr}
2 & 1 & 1 & 7 \\
0 & 1 & -1 & -3 \\
0 & -1 & 1 & 3
\end{array}\right]} \\
& \xrightarrow{\substack{R 1 \rightarrow R 1-R 2 \\
R 3 \rightarrow R 3+R 2}}\left[\begin{array}{rrrr}
2 & 0 & 2 & 10 \\
0 & 1 & -1 & -3 \\
0 & 0 & 0 & 0
\end{array}\right] \xrightarrow{R 1 \rightarrow \frac{1}{2} R 1}\left[\begin{array}{rrrr}
1 & 0 & 1 & 5 \\
0 & 1 & -1 & -3 \\
0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

giving general solution

$$
\left\{\begin{array}{l}
x_{1}=5-x_{3} \\
x_{2}=-3+x_{3} \\
x_{3} \text { free }
\end{array}\right.
$$

This gives the set of all least-squares solutions.
7. The least-squares error is the distance:

$$
\|\mathbf{b}-A \hat{\mathbf{x}}\|=\left\|\left[\begin{array}{r}
3 \\
1 \\
-4 \\
2
\end{array}\right]-\left[\begin{array}{rr}
1 & -2 \\
-1 & 2 \\
0 & 3 \\
2 & 5
\end{array}\right]\left[\begin{array}{r}
4 / 3 \\
-1 / 3
\end{array}\right]\right\|=\left\|\left[\begin{array}{r}
3 \\
1 \\
-4 \\
2
\end{array}\right]-\left[\begin{array}{r}
2 \\
-2 \\
-1 \\
1
\end{array}\right]\right\|=\left\|\left[\begin{array}{r}
1 \\
3 \\
-3 \\
1
\end{array}\right]\right\|=\sqrt{20}
$$

## Exercises 7.1 (p. 448)

Assignment: Do \#25, 26, 1-6, 13, 17, 23, 27, 28, 30, 34
25. (a) True. (Theorem 7.2, p. 445)
(b) True. The vectors are from distinct eigenspaces, so they are orthogonal by Theorem 7.1.
(c) False. It has $n$ real eigenvalues counting multiplicities, but it may not have $n$ distinct eigenvalues. (We did an example in class of a $3 \times 3$ symmetric matrix with only two distinct eigenvalues.)
(d) True. (p. 447) It projects any vector onto the subspace spanned by $\mathbf{v}$.
26. (a) True. (Theorem 7.2, p. 445)
(b) True. By definition, $B$ is orthogonally diagonalizable. Therefore, it is symmetric by Theorem 7.2.
(c) False. Orthogonal matrices need not be symmetric, and only symmetric matrices are orthogonally diagonalizable.
(d) True. (Theorem 7.3, p. 446)

1-6. Only the matrices in problems 1 and 4 are symmetric.
13. Finding the eigenvalues,

$$
\operatorname{det}(A-\lambda I)=(3-\lambda)^{2}-1=\lambda^{2}-6 \lambda+8=(\lambda-4)(\lambda-2)
$$

giving eigenvalues $\lambda=2,4$.
For $\lambda=2$, the eigenspace is given by the solution set associated with the augmented matrix:

$$
\left[\begin{array}{lll}
1 & 1 & 0 \\
1 & 1 & 0
\end{array}\right] \xrightarrow{R 2 \rightarrow R 2-R 1}\left[\begin{array}{lll}
1 & 1 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

which has general solution

$$
\left\{\begin{array}{l}
x_{1}=-x_{2} \\
x_{2} \text { free }
\end{array}\right.
$$

and parametric vector form

$$
\mathbf{x}=\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{r}
-x_{2} \\
x_{2}
\end{array}\right]=x_{2}\left[\begin{array}{r}
-1 \\
1
\end{array}\right], \quad x_{2} \text { free }
$$

The basis for the eigenspace is

$$
\left\{\left[\begin{array}{r}
-1 \\
1
\end{array}\right]\right\}
$$

For $\lambda=4$, the eigenspace is given by

$$
\left[\begin{array}{rrr}
-1 & 1 & 0 \\
1 & -1 & 0
\end{array}\right] \xrightarrow{R 2 \rightarrow R 2+R 1}\left[\begin{array}{rrr}
-1 & 1 & 0 \\
0 & 0 & 0
\end{array}\right] \xrightarrow{R 1 \rightarrow-R 1}\left[\begin{array}{rrr}
1 & -1 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

which has general solution

$$
\left\{\begin{array}{l}
x_{1}=x_{2} \\
x_{2} \text { free }
\end{array}\right.
$$

and parametric vector form

$$
\mathbf{x}=\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
x_{2} \\
x_{2}
\end{array}\right]=x_{2}\left[\begin{array}{l}
1 \\
1
\end{array}\right], \quad x_{2} \text { free }
$$

The basis for the eigenspace is

$$
\left\{\left[\begin{array}{l}
1 \\
1
\end{array}\right]\right\}
$$

There are no eigenbases with more than one vector, so there's no need to orthogonalize anything.
We need only normalize the vectors:

$$
\begin{array}{ll}
(\lambda=2) & \frac{1}{\left\|\left[\begin{array}{r}
-1 \\
1
\end{array}\right]\right\|}\left[\begin{array}{r}
-1 \\
1
\end{array}\right]=\frac{1}{\sqrt{(-1)^{2}+1^{2}}}\left[\begin{array}{r}
-1 \\
1
\end{array}\right]=\frac{1}{\sqrt{2}}\left[\begin{array}{r}
-1 \\
1
\end{array}\right]=\left[\begin{array}{r}
-1 / \sqrt{2} \\
1 / \sqrt{2}
\end{array}\right] \\
(\lambda=4) & \frac{1}{\left\|\left[\begin{array}{l}
1 \\
1
\end{array}\right]\right\|}\left[\begin{array}{l}
1 \\
1
\end{array}\right]=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
1 \\
1
\end{array}\right]=\left[\begin{array}{l}
1 / \sqrt{2} \\
1 / \sqrt{2}
\end{array}\right]
\end{array}
$$

Put this orthonormal set in the columns of $P$ and the corresponding eigenvalues along the main diagonal of $D$ to get:

$$
P=\left[\begin{array}{rr}
-1 / \sqrt{2} & 1 / \sqrt{2} \\
1 / \sqrt{2} & 1 / \sqrt{2}
\end{array}\right] \quad D=\left[\begin{array}{ll}
2 & 0 \\
0 & 4
\end{array}\right]
$$

17. The eigenvalues are given in the problem: 5,2 , and -2 .

For $\lambda=5$, the eigenspace has basis given by

$$
\left[\begin{array}{rrrr}
-4 & 1 & 3 & 0 \\
1 & -2 & 1 & 0 \\
3 & 1 & -4 & 0
\end{array}\right] \xrightarrow{R 1 \leftrightarrow R 2}\left[\begin{array}{rrrr}
1 & -2 & 1 & 0 \\
-4 & 1 & 3 & 0 \\
3 & 1 & -4 & 0
\end{array}\right] \xrightarrow{\substack{R 2 \rightarrow R 2+4 R 1 \\
R 3 \rightarrow R 3-3 R 1}}\left[\begin{array}{rrrr}
1 & -2 & 1 & 0 \\
0 & -7 & 7 & 0 \\
0 & 7 & -7 & 0
\end{array}\right]
$$

$$
\xrightarrow{R 2 \rightarrow-\frac{1}{7} R 2}\left[\begin{array}{rrrr}
1 & -2 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 7 & -7 & 0
\end{array}\right] \xrightarrow{\substack{R 1 \rightarrow R 1+2 R 2 \\
R 3 \rightarrow R 3-7 R 2}}\left[\begin{array}{rrrr}
1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

giving general solution

$$
\left\{\begin{array}{l}
x_{1}=x_{3} \\
x_{2}=x_{3} \\
x_{3} \text { free }
\end{array}\right.
$$

and parametric vector form

$$
\mathbf{x}=\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=\left[\begin{array}{l}
x_{3} \\
x_{3} \\
x_{3}
\end{array}\right]=x_{3}\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right], \quad x_{3} \text { free }
$$

So, the basis is:

$$
\left\{\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]\right\}
$$

For $\lambda=2$, the eigenspace has basis given by

$$
\left[\begin{array}{rrrr}
-1 & 1 & 3 & 0 \\
1 & 1 & 1 & 0 \\
3 & 1 & -1 & 0
\end{array}\right] \xrightarrow{\substack{R 2 \rightarrow R 2+R 1 \\
R 3 \rightarrow R 3+3 R 1}}\left[\begin{array}{rrrr}
-1 & 1 & 3 & 0 \\
0 & 2 & 4 & 0 \\
0 & 4 & 8 & 0
\end{array}\right] \xrightarrow{R 2 \rightarrow \frac{1}{2} R 2}\left[\begin{array}{rrrr}
-1 & 1 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 4 & 8 & 0
\end{array}\right]
$$

giving general solution

$$
\left\{\begin{array}{l}
x_{1}=x_{3} \\
x_{2}=-2 x_{3} \\
x_{3} \text { free }
\end{array}\right.
$$

and parametric vector form

$$
\mathbf{x}=\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=\left[\begin{array}{r}
x_{3} \\
-2 x_{3} \\
x_{3}
\end{array}\right]=x_{3}\left[\begin{array}{r}
1 \\
-2 \\
1
\end{array}\right], \quad x_{3} \text { free }
$$

So, the basis is:

$$
\left\{\left[\begin{array}{r}
1 \\
-2 \\
1
\end{array}\right]\right\}
$$

For $\lambda=-2$, the eigenspace has basis given by

$$
\begin{aligned}
& {\left[\begin{array}{llll}
3 & 1 & 3 & 0 \\
1 & 5 & 1 & 0 \\
3 & 1 & 3 & 0
\end{array}\right] \xrightarrow{R 1 \leftrightarrow R 2}\left[\begin{array}{llll}
1 & 5 & 1 & 0 \\
3 & 1 & 3 & 0 \\
3 & 1 & 3 & 0
\end{array}\right] \xrightarrow{R 3 \rightarrow R 3-R 2}\left[\begin{array}{llll}
1 & 5 & 1 & 0 \\
3 & 1 & 3 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]} \\
& \xrightarrow{R 2 \rightarrow R 2-3 R 1}\left[\begin{array}{rrrr}
1 & 5 & 1 & 0 \\
0 & -14 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right] \xrightarrow{R 2 \rightarrow-\frac{1}{14} R 2}\left[\begin{array}{llll}
1 & 5 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right] \xrightarrow{R 1 \rightarrow R 1-5 R 2}\left[\begin{array}{llll}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
\end{aligned}
$$

giving general solution

$$
\left\{\begin{array}{l}
x_{1}=-x_{3} \\
x_{2}=0 \\
x_{3} \text { free }
\end{array}\right.
$$

and parametric vector form

$$
\mathbf{x}=\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=\left[\begin{array}{r}
-x_{3} \\
0 \\
x_{3}
\end{array}\right]=x_{3}\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right], \quad x_{3} \text { free }
$$

So, the basis is:

$$
\left\{\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right]\right\}
$$

Again, none of the bases contains more than one vector, so there is no need to orthogonalize any vectors.
Normalizing the vectors, we get

$$
\begin{array}{ll}
(\lambda=5) & \frac{1}{\left\|\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]\right\|}\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]=\frac{1}{\sqrt{3}}\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]=\left[\begin{array}{l}
1 / \sqrt{3} \\
1 / \sqrt{3} \\
1 / \sqrt{3}
\end{array}\right] \\
(\lambda=2) & \left.\frac{1}{\left\|\left[\begin{array}{r}
1 \\
-2 \\
1
\end{array}\right]\right\|\left[\begin{array}{r}
1 \\
-2 \\
1
\end{array}\right]=\frac{1}{\sqrt{6}}\left[\begin{array}{r}
1 \\
-2 \\
1
\end{array}\right]=\left[\begin{array}{r}
1 / \sqrt{6} \\
-2 / \sqrt{6} \\
1 / \sqrt{6}
\end{array}\right]} \begin{array}{ll}
(\lambda=-2) & \frac{1}{\left\|\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right]\right\|\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right]=\frac{1}{\sqrt{2}}\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right]=\left[\begin{array}{r}
-1 / \sqrt{2} \\
0 \\
1 / \sqrt{2}
\end{array}\right]}
\end{array} .=\begin{array}{l}
\|
\end{array}\right]
\end{array}
$$

giving matrices:

$$
P=\left[\begin{array}{rrr}
1 / \sqrt{3} & 1 / \sqrt{6} & -1 / \sqrt{2} \\
1 / \sqrt{3} & -2 / \sqrt{6} & 0 / \sqrt{2} \\
1 / \sqrt{3} & 1 / \sqrt{6} & 1 / \sqrt{2}
\end{array}\right] \quad D=\left[\begin{array}{rrr}
5 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -2
\end{array}\right]
$$

23. First, we will verify that 5 is an eigenvalue. We could calculate $\operatorname{det}(A-5 I)$ and show that it is zero. However, we are going to need to find a basis for the eigenspace associated with $\lambda=5$ anyway, so we might as well solve the homogeneous system $(A-5 I) \mathbf{x}=\mathbf{0}$ and show that it has nontrivial solutions:

$$
\left[\begin{array}{rrrr}
-2 & 1 & 1 & 0 \\
1 & -2 & 1 & 0 \\
1 & 1 & -2 & 0
\end{array}\right] \longrightarrow \quad \ldots \quad\left[\begin{array}{rrrr}
1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

giving

$$
\left\{\begin{array}{l}
x_{1}=x_{3} \\
x_{2}=x_{3} \\
x_{3} \text { free }
\end{array}\right.
$$

and

$$
\mathbf{x}=\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=x_{3}\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right], \quad x_{3} \text { free }
$$

so the basis is

$$
\left\{\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]\right\}
$$

This shows that 5 is an eigenvalue.
Now, we will show that $\mathbf{v}$ is an eigenvector:

$$
A \mathbf{v}=\left[\begin{array}{lll}
3 & 1 & 1 \\
1 & 3 & 1 \\
1 & 1 & 3
\end{array}\right]\left[\begin{array}{r}
-1 \\
1 \\
0
\end{array}\right]=\left[\begin{array}{r}
-2 \\
2 \\
0
\end{array}\right]=2\left[\begin{array}{r}
-1 \\
1 \\
0
\end{array}\right]=2 \mathbf{v}
$$

so it an eigenvector for eigenvalue 2. To get a basis for the eigenspace associated with $\lambda=2$, we solve the following system:

$$
\left[\begin{array}{llll}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0
\end{array}\right] \xrightarrow{\substack{R 2 \rightarrow R 2-R 1 \\
R 3 \rightarrow R 3-R 1}}\left[\begin{array}{llll}
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

giving

$$
\left\{\begin{array}{l}
x_{1}=-x_{2}-x_{3} \\
x_{2}, x_{3} \text { free }
\end{array}\right.
$$

and

$$
\mathbf{x}=\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=x_{2}\left[\begin{array}{r}
-1 \\
1 \\
0
\end{array}\right]+x_{3}\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right], \quad x_{2}, x_{3} \text { free }
$$

so the basis is

$$
\left\{\left[\begin{array}{r}
-1 \\
1 \\
0
\end{array}\right],\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right]\right\}
$$

Observe that we now have our three (linearly independent) eigenvectors, so there are no more eigenvalues or eigenbases to find.
However, the second basis contains two vectors and must be orthogonalized using GramSchmidt. This gives

$$
\begin{aligned}
& \mathbf{v}_{1}=\left[\begin{array}{r}
-1 \\
1 \\
0
\end{array}\right] \\
& \mathbf{v}_{2}=\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right]-\frac{\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right] \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1}=\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right]-\frac{\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right] \cdot\left[\begin{array}{r}
-1 \\
1 \\
0
\end{array}\right]}{\left[\begin{array}{r}
-1 \\
1 \\
0
\end{array}\right] \cdot\left[\begin{array}{r}
-1 \\
1 \\
1 \\
0
\end{array}\right]}\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right]-\frac{1}{2}\left[\begin{array}{r}
-1 \\
1 \\
0
\end{array}\right]=\left[\begin{array}{r}
-1 / 2 \\
-1 / 2 \\
1
\end{array}\right]
\end{aligned}
$$

Our collection of three orthogonal vectors is now:

$$
\left\{\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right],\left[\begin{array}{r}
-1 \\
1 \\
0
\end{array}\right],\left[\begin{array}{r}
-1 / 2 \\
-1 / 2 \\
1
\end{array}\right]\right\}
$$

associated with eigenvalues 5,2 , and 2 respectively.
Normalizing the vectors gives:

$$
\left\{\left[\begin{array}{l}
1 / \sqrt{3} \\
1 / \sqrt{3} \\
1 / \sqrt{3}
\end{array}\right],\left[\begin{array}{r}
-1 / \sqrt{2} \\
1 / \sqrt{2} \\
0
\end{array}\right],\left[\begin{array}{r}
-1 / \sqrt{6} \\
-1 / \sqrt{6} \\
2 / \sqrt{6}
\end{array}\right]\right\}
$$

The final matrices giving $A=P D P^{-1}$ are

$$
P=\left[\begin{array}{rrr}
1 / \sqrt{3} & -1 / \sqrt{2} & -1 / \sqrt{6} \\
1 / \sqrt{3} & 1 / \sqrt{2} & -1 / \sqrt{6} \\
1 / \sqrt{3} & 0 & 2 / \sqrt{6}
\end{array}\right] \quad D=\left[\begin{array}{lll}
5 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{array}\right]
$$

(with $P^{-1}=P^{T}$ ).
27. (a) $\left(B^{T} A B\right)^{T}=B^{T} A^{T}\left(B^{T}\right)^{T}=B^{T} A^{T} B=B^{T} A B$;
(b) $\left(B^{T} B\right)^{T}=B^{T}\left(B^{T}\right)^{T}=B^{T} B$;
(c) $\left(B B^{T}\right)^{T}=\left(B^{T}\right)^{T} B^{T}=B B^{T}$.
28. $(A \mathbf{x}) \cdot \mathbf{y}=(A \mathbf{x})^{T} \mathbf{y}=\mathbf{x}^{T} A^{T} \mathbf{y}=\mathbf{x}^{T} A \mathbf{y}=\mathbf{x} \cdot(A \mathbf{y})$
30. By Theorem 7.2, $A$ and $B$ are symmetric. It is enough to show that this implies $A B$ is symmetric: then, another application of Theorem 7.2 will show that $A B$ is orthogonally diagonalizable. To show that $A B$ is symmetric:

$$
(A B)^{T}=B^{T} A^{T}=B A=A B
$$

(where the second last equality follows from $A$ and $B$ being symmetric and the last equality was given in the problem).
34.

$$
\begin{aligned}
A & =\lambda_{1} \mathbf{u}_{1} \mathbf{u}_{1}^{T}+\lambda_{2} \mathbf{u}_{2} \mathbf{u}_{2}^{T}+\lambda_{3} \mathbf{u}_{3} \mathbf{u}_{3}^{T} \\
= & \left.7\left[\begin{array}{rr}
1 / \sqrt{2} \\
0 \\
1 / \sqrt{2}
\end{array}\right] \begin{array}{rr}
1 / \sqrt{2} & 0 \\
1 / \sqrt{2}
\end{array}\right]+7\left[\begin{array}{r}
-1 / \sqrt{18} \\
4 / \sqrt{18} \\
1 / \sqrt{18}
\end{array}\right]\left[\begin{array}{ll}
-1 / \sqrt{18} & 4 / \sqrt{18} \\
1 / \sqrt{18}
\end{array}\right] \\
& -2\left[\begin{array}{r}
-2 / 3 \\
-1 / 3 \\
2 / 3
\end{array}\right]\left[\begin{array}{rrr}
-2 / 3 & -1 / 3 & 2 / 3
\end{array}\right] \\
& =7\left[\begin{array}{rrr}
1 / 2 & 0 & 1 / 2 \\
0 & 0 & 0 \\
1 / 2 & 0 & 1 / 2
\end{array}\right]+7\left[\begin{array}{rrr}
1 / 18 & -4 / 18 & -1 / 18 \\
-4 / 18 & 16 / 18 & 4 / 18 \\
-1 / 18 & 4 / 18 & 1 / 18
\end{array}\right]-2\left[\begin{array}{rrr}
4 / 9 & 2 / 9 & -4 / 9 \\
2 / 9 & 1 / 9 & -2 / 9 \\
-4 / 9 & -2 / 9 & 4 / 9
\end{array}\right]
\end{aligned}
$$

## Exercises 7.2 (p. 457)

Assignment: Do \#21, 22, 1, 3, 7, 25
21. (a) True. (p. 450)
(b) True.
(c) True. By the discussion preceding Theorem 7.4, the symmetric matrix $A$ can be orthogonally diagonalized $A=P D P^{-1}$, and the columns of $P$ (which are orthonormal eigenvectors of $A$ ) are the principal axes.
(d) False. It satisfies $Q(\mathbf{x})>0$ for all $\mathbf{x} \neq \mathbf{0}$, but of course $Q(\mathbf{0})=0$ for all quadratic forms.
(e) True. (Theorem 7.5, p. 456)
(f) True. (p. 457)
22. (a) True. (p. 450)
(b) False. It doesn't work for any orthogonal matrix $P$ only those that arise in orthogonal diagonalizations of $A$.
(c) False. Depending on $A$ and $c$, it can also be empty, a point, a line, or two lines.
(d) False. It's one that takes on both positive and negative values.
(e) True. (Theorem 7.5, p. 456)

1. (a)

$$
\begin{aligned}
& Q(\mathbf{x})=\mathbf{x}^{T} A \mathbf{x}=\left[\begin{array}{ll}
x_{1} & x_{2}
\end{array}\right]\left[\begin{array}{cc}
5 & 1 / 3 \\
1 / 3 & 1
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right] \\
&=\left[\begin{array}{ll}
5 x_{1}+\frac{1}{3} x_{2} & \frac{1}{3} x_{1}+x_{2}
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=5 x_{1}^{2}+\frac{2}{3} x_{1} x_{2}+x_{2}^{2}
\end{aligned}
$$

Alternatively, we could have just observed that the diagonal elements 5 and 1 contribute $5 x_{1}^{2}+x_{2}^{2}$ to the sum while the off-diagonal element gets doubled to produce $(2 / 3) x_{1} x_{2}$.
(b) $Q\left(\left[\begin{array}{l}6 \\ 1\end{array}\right]\right)=5(6)^{2}+\frac{2}{3}(6)(1)+(1)^{2}=180+4+1=185$
(c) $Q\left(\left[\begin{array}{l}1 \\ 3\end{array}\right]\right)=5(1)^{2}+\frac{2}{3}(1)(3)+(3)^{2}=5+2+9=16$
3. (a) The diagonal elements are given by the coefficients on the terms $10 x_{1}^{2}$ and $-3 x_{2}^{2}$. The off-diagonal element is half the coefficient on the cross-product term $-6 x_{1} x_{2}$, so we have

$$
A=\left[\begin{array}{rr}
10 & -3 \\
-3 & -3
\end{array}\right]
$$

(b) The diagonal elements are given by $5 x_{1}^{2}+0 x_{2}^{2}$ and the off-diagonal element is half of the coefficient of $3 x_{1} x_{2}$ :

$$
A=\left[\begin{array}{cc}
5 & 3 / 2 \\
3 / 2 & 0
\end{array}\right]
$$

7. The matrix $A$ is given by

$$
A=\left[\begin{array}{ll}
1 & 5 \\
5 & 1
\end{array}\right]
$$

We need an orthogonal diagonalization (to calculate $P$ ). The characteristic polynomial

$$
\operatorname{det}(A-\lambda I)=(1-\lambda)^{2}-25=\lambda^{2}-2 \lambda-24=(\lambda-6)(\lambda+4)
$$

gives eigenvalues -4 and 6 . For $\lambda=-4$, the usual computation gives an eigenbasis $\left\{\left[\begin{array}{r}-1 \\ 1\end{array}\right]\right\}$. For $\lambda=6$, we get an eigenbasis $\left\{\left[\begin{array}{l}1 \\ 1\end{array}\right]\right\}$. These are orthogonal but not normal. Normalizing, we get the columns of matrix $P$, and the matrices $P$ and $D$ are:

$$
P=\left[\begin{array}{rr}
-1 / \sqrt{2} & 1 / \sqrt{2} \\
1 / \sqrt{2} & 1 / \sqrt{2}
\end{array}\right] \quad D=\left[\begin{array}{rr}
-4 & 0 \\
0 & 6
\end{array}\right]
$$

By the Principal Axis Theorem (and the discussion preceding it), $P$ is the change-of-variable matrix such that the change of variables $\mathbf{x}=P \mathbf{y}$ gives the new quadratic form

$$
\mathbf{y}^{T} D \mathbf{y}=-4 y_{1}^{2}+6 y_{2}^{2}
$$

with no cross-product term.
25. (a) We must show that $\mathbf{x}^{T}\left(B^{T} B\right) \mathbf{x} \geq 0$ for all $\mathbf{x}$. However, we see that

$$
\mathbf{x}^{T}\left(B^{T} B\right) \mathbf{x}=\left(\mathbf{x}^{T} B^{T}\right)(B \mathbf{x})=(B \mathbf{x})^{T}(B \mathbf{x})=(B \mathbf{x}) \cdot(B \mathbf{x})
$$

and this last expression is $\geq 0$ by the properties of inner products. (In fact, it equals $\|B \mathbf{x}\|^{2}$.)
(b) If, in addition, $B$ is $n \times n$ and invertible, then, as above

$$
\mathbf{x}^{T}\left(B^{T} B\right) \mathbf{x}=\|B \mathbf{x}\|^{2} \geq 0
$$

However, the only way this squared length can equal zero is if $B \mathbf{x}=\mathbf{0}$ (by the properties of length). Since $B$ is invertible, the only solution of this homogenous equation is $\mathbf{x}=\mathbf{0}$. Therefore, for all non-zero vectors $\mathbf{x}, B \mathbf{x}$ is nonzero, so the squared-length is $>0$.

