Homework Set #11 Solutions

Corrections: (Dec. 11) 6.3 #7

Exercises 5.4 (p. 327)

Assignment: Do #2, 4, 8, 12, 16, 20, 22

2. This matrix is

$$M = \big[[T(\mathbf{d}_1)]_{\mathfrak{B}} \quad [T(\mathbf{d}_2)]_{\mathfrak{B}}\big]$$

The first column is the coefficient vector of $T(\mathbf{d}_1)$ with respect to basis $\mathfrak{B}$, but since

$$T(\mathbf{d}_1) = 2\mathbf{b}_1 - 3\mathbf{b}_2$$

this vector is simply $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Similarly, the second column is $\begin{bmatrix} -4 \\ 5 \end{bmatrix}$, so the matrix is

$$M = \begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$$

4. The terminology is a bit confusing here: the desired matrix is the matrix for T relative to $\mathfrak{B}$ and $\mathfrak{E}$ where $\mathfrak{B}$ is the given basis for V and $\mathfrak{E}$ is the standard basis for $\mathbb{R}^2$. Therefore, we want

$$M = \big[[T(\mathbf{b}_1)]_{\mathfrak{E}} \quad [T(\mathbf{b}_2)]_{\mathfrak{E}} \quad [T(\mathbf{b}_3)]_{\mathfrak{E}}\big]$$

To calculate the first column, we see that

$$T(\mathbf{b}_1) = T(1\mathbf{b}_1 + 0\mathbf{b}_2 + 0\mathbf{b}_3) = \begin{bmatrix} 2(1) - 4(0) + 5(0) \\ -(0) + 3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

The first column of M is the coordinate vector of this vector with respect to the standard basis $\mathfrak{E}$. However, the coordinate vector of any vector with respect to the standard basis is itself (see Example 2 on page 241), so the first column of M is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Similarly, the second column is

$$[T(\mathbf{b}_2)]_{\mathfrak{E}} = T(\mathbf{b}_2) = T(0\mathbf{b}_1 + 1\mathbf{b}_2 + 0\mathbf{b}_3) = \begin{bmatrix} 2(0) - 4(1) + 5(0) \\ -(1) + 3(0) \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

and the third column is

$$[T(\mathbf{b}_3)]_{\mathfrak{C}} = T(\mathbf{b}_3) = T(0\mathbf{b}_1 + 0\mathbf{b}_2 + 1\mathbf{b}_3) = \begin{bmatrix} 2(0) - 4(0) + 5(1) \\ -(0) + 3(1) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

so

$$M = \begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}$$

8. We know that for any $\mathbf{v} \in V$, we have

$$[T(\mathbf{v})]_{\mathfrak{B}} = [T]_{\mathfrak{B}}[\mathbf{v}]_{\mathfrak{B}}$$

However

$$[3\mathbf{b}_1 - 4\mathbf{b}_2]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$$

so we can calculate

$$[T(\mathbf{v})]_{\mathfrak{B}} = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ -20 \\ 11 \end{bmatrix}$$

which implies

$$T(\mathbf{v}) = 24\mathbf{b}_1 - 20\mathbf{b}_2 + 11\mathbf{b}_3$$

12. We must find

$$[T]_{\mathfrak{B}} = \big[[T(\mathbf{b}_1)]_{\mathfrak{B}} \quad [T(\mathbf{b}_2)]_{\mathfrak{B}}\big] = \big[[A\mathbf{b}_1]_{\mathfrak{B}} \quad [A\mathbf{b}_2]_{\mathfrak{B}}\big]$$

For the first column,

$$A\mathbf{b}_1 = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

To express it in $\mathfrak{B}$-coordinates, we must find weights such that $\begin{bmatrix} 5 \\ 0 \end{bmatrix} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 = x_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. In other words, we must solve the equation

$$\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

The usual method yields the unique solution $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

For the second column,

$$A\mathbf{b}_1 = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

and solving the equation

$$\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

gives the unique solution $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Thus, the matrix is

$$M = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

16. We must diagonalize A. Its characteristic equation is

$$(2-\lambda)(3-\lambda) - 6 = \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$$

yields the eigenvalues 0 and 5. The usual method gives a basis for the null space of $A - 0I$, namely $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$, and a basis for the null space of $A - 5\lambda$, namely $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$. This gives a diagonalization $A = PDP^{-1}$ with

$$P = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

By Theorem 5.8, the basis

$$\mathfrak{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

satisfies $[T]_{\mathfrak{B}} = D$, a diagonal matrix.

20. By definition, A and B similar implies there exists an invertible P such that $A = PBP^{-1}$. Therefore,

$$A^2 = (PBP^{-1})(PBP^{-1}) = PB(P^{-1}P)BP^{-1} = PBBP^{-1} = PB^2P^{-1}$$

and so, by definition, A^2 is similar to B^2.

22. By definition, A diagonalizable implies there exists an invertible P and a diagonal D such that $A = PDP^{-1}$. If A and B are similar, then there exists an invertible matrix, say Q, such that $B = QAQ^{-1}$. But, then,

$$B = QAQ^{-1} = QPDP^{-1}Q^{-1}$$

Since P and Q invertible implies QP invertible with $(QP)^{-1} = P^{-1}Q^{-1}$, there exists an invertible matrix (specifically $R = QP$) such that $B = RDR^{-1}$. That is, by definition, B is diagonalizable.

Exercises 6.1 (p. 376)

Assignment: Do #19, 20, 1, 5, 8, 12, 13, 26

19. (a) True. (p. 370)

(b) True. (Theorem 5.1, p. 370)

(c) True. (p. 373)

(d) False. For any matrix, $\operatorname{Col} A$ and $\operatorname{Nul} A^T$ are orthogonal, but we don't have $\operatorname{Col} A$ and $\operatorname{Nul} A$ orthogonal in general, even if A is square. For example, the matrix $A = \left[\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right]$ has $(1,1)$ in its column space and $(0,1)$ in its null space, and these vectors are not orthogonal.

(e) True. (p. 374)

20. (a) True, since $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$.

(b) False. Instead, $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$.

(c) True. (p. 374)

(d) True. (Theorem 5.2, p. 374)

(e) True. (Theorem 5.3, p. 375)

1. $\mathbf{u}\cdot\mathbf{u} = (-1)(-1) + 2(2) = 5 \qquad \mathbf{v}\cdot\mathbf{u} = 4(-1) + 6(2) = 8 \qquad \dfrac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}} = \dfrac{8}{5}$

5.

$$\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{8}{4^2+6^2}\right)\mathbf{v} = \frac{8}{52}\begin{bmatrix}4\\6\end{bmatrix} = \begin{bmatrix}8/13\\12/13\end{bmatrix}$$

8. $\|\mathbf{x}\| = \sqrt{\mathbf{x}\cdot\mathbf{x}} = \sqrt{6^2 + (-2)^2 + 3^2} = \sqrt{36+4+9} = \sqrt{49} = 7$

12. Normalizing the vector:

$$\frac{1}{\left\|\begin{bmatrix}8/3\\2\end{bmatrix}\right\|}\begin{bmatrix}8/3\\2\end{bmatrix} = \left(\frac{1}{\sqrt{64/9+4}}\right)\begin{bmatrix}8/3\\2\end{bmatrix} = \left(\frac{1}{\sqrt{100/9}}\right)\begin{bmatrix}8/3\\2\end{bmatrix} = \left(\frac{1}{10/3}\right)\begin{bmatrix}8/3\\2\end{bmatrix} = \frac{3}{10}\begin{bmatrix}8/3\\2\end{bmatrix} = \begin{bmatrix}4/5\\3/5\end{bmatrix}$$

13.

$$\operatorname{dist}\left(\begin{bmatrix}10\\-3\end{bmatrix}, \begin{bmatrix}-1\\-5\end{bmatrix}\right) = \left\|\begin{bmatrix}10\\-3\end{bmatrix} - \begin{bmatrix}-1\\-5\end{bmatrix}\right\| = \left\|\begin{bmatrix}11\\2\end{bmatrix}\right\| = \sqrt{11^2+2^2} = \sqrt{125} = 5\sqrt{5}$$

26. We can rewrite the equation $\mathbf{u}\cdot\mathbf{x} = 0$ as the homogeneous matrix equation $\mathbf{u}^T\mathbf{x} = \mathbf{0}$. Therefore, by Theorem 4.2, the solution set of this equation (which is the set W) is a subspace of $\mathbb{R}^3$. Since the dimension of the column space of the matrix $\mathbf{u}^T$ is 1, by the Rank Theorem, the dimension of the null space W is 2. A 2-dimensional subspace of $\mathbb{R}^3$ is a plane through the origin, so W is the plane through the origin perpendicular to $\mathbf{u}$.

Exercises 6.2 (p. 386)

Assignment: Do #23, 24, 1, 3, 9, 11, 15, 20

23. (a) True. Most linearly independent sets in $\mathbb{R}^n$ *aren't* orthogonal sets.

(b) True. Theorem 6.5 can be used to calculate the weights.

(c) False. The normalized vectors are always orthogonal if the original vectors were.

(d) False. A *square* matrix with orthonormal columns is an orthogonal matrix.

(e) False. $\|\mathbf{y} - \hat{\mathbf{y}}\|$ gives the distance from $\mathbf{y}$ to L.

24. (a) True. Orthogonal sets can contain the zero vector, and such a set is linearly dependent.

(b) False. Such a set is orthogonal, but the vectors must be unit length (must have $\mathbf{u}_i \cdot \mathbf{u}_i = 1$ for all i) in order to be ortho*normal*.

(c) True. (p. 385)

(d) True. (p. 381)

(e) True. (p. 385)

1. Since
$$\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix} = (-1)(3) + 4(-4) + (-3)(-7) = 2 \neq 0$$
this set is *not* orthogonal.

3. Since
$$\begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = -6(3) + (-3)(1) + 9(-1) = -18 - 3 - 9 = -30 \neq 0$$
this set is *not* orthogonal.

9. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = 1(-1) + 0(4) + 1(1) = 0$, $\mathbf{u}_1 \cdot \mathbf{u}_3 = 1(2) + 0(1) + 1(-2) = 0$, and $\mathbf{u}_2 \cdot \mathbf{u}_3 = -1(2) + 4(1) + 1(-2) = 0$, the set is orthogonal. Because it contains no zero vectors, it is linearly independent, and a linearly independent set of 3 vectors in $\mathbb{R}^3$ is a basis for $\mathbb{R}^3$. We can use Theorem 6.5 to write
$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{x} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3$$

which gives

$$\mathbf{x} = \frac{\begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= \frac{5}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{-27}{18} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + \frac{18}{9} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

11. This is simply

$$\frac{\begin{bmatrix} 1 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 2 \end{bmatrix}}{\begin{bmatrix} -4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 2 \end{bmatrix}} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \frac{10}{20} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

15. This is the length of:

$$\mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 6 \end{bmatrix}}{\begin{bmatrix} 8 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 6 \end{bmatrix}} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{30}{100} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$$

which is

$$\|\mathbf{y} - \hat{\mathbf{y}}\| = \sqrt{9/25 + 16/25} = \sqrt{25/25} = 1$$

20. Note that this set is orthogonal. The length of the first vector is

$$\sqrt{(-2/3)^2 + (1/3)^2 + (2/3)^2} = \sqrt{4/9 + 1/9 + 4/9} = \sqrt{9/9} = 1$$

so it is already a unit vector. The length of the second vector is

$$\sqrt{(1/3)^2 + (2/3)^2} = \sqrt{1/9 + 4/9} = \sqrt{5/9} = \sqrt{5}/3$$

Normalizing it gives

$$\frac{1}{\sqrt{5}/3} \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} = \frac{3}{\sqrt{5}} \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$$

giving the orthonormal set

$$\left\{ \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \right\}$$

Exercises 6.3 (p. 395)

Assignment: Do #21, 22, 2, 3, 7, 11, 18, 19

21. (a) True. (p. 374)

(b) True. (Theorem 6.8, p. 390)

(c) False. (p. 391)

(d) True. (p. 392)

(e) True. (p. 394)

22. (a) True. Suppose $\mathbf{v} \in W$. If $\mathbf{v} \in W^{\perp}$, then it is orthogonal to every vector in W. In particular, it must be orthogonal to itself: $\mathbf{v} \cdot \mathbf{v} = 0$. But this is only true for the zero vector, so $\mathbf{v} = \mathbf{0}$.

(b) True. (p. 390)

(c) True. (Theorem 6.8, p. 390)

(d) False. It's given by $\text{proj}_W \mathbf{y}$.

(e) False. Instead, $UU^T\mathbf{y} = \text{proj}_W \mathbf{y}$ for all $\mathbf{y} \in \mathbb{R}^n$ and $U^TU\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^p$.

2. These four nonzero, orthogonal vectors must form a basis for $\mathbb{R}^4$, so by Theorem 6.5, we can write

$$\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \underbrace{\frac{\mathbf{v} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{v} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 + \frac{\mathbf{v} \cdot \mathbf{u}_4}{\mathbf{u}_4 \cdot \mathbf{u}_4}\mathbf{u}_4}_{\mathbf{z}}$$

The first of these terms is easily calculated:

$$\frac{\begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \frac{14}{7} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

It is unnecessary to calculate the sum $\mathbf{z}$ of the other three terms explicitly. Instead, we write:

$$\mathbf{z} = \mathbf{v} - \hat{\mathbf{v}} = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix}$$

is the desired sum.

3. By Theorem 6.8,

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \frac{\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

7. By Theorem 6.8,

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \frac{\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}\cdot\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}}{\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}}\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

$$= \frac{0}{14}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{28}{42}\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix}$$

and

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}$$

so the desired sum is

$$\mathbf{y} = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} + \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}$$

11. By the Best Approximation Theorem, this is

$$\hat{\mathbf{y}} = \frac{\begin{bmatrix}3\\1\\5\\1\end{bmatrix}\cdot\begin{bmatrix}3\\1\\-1\\1\end{bmatrix}}{\begin{bmatrix}3\\1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}3\\1\\-1\\1\end{bmatrix}}\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} + \frac{\begin{bmatrix}3\\1\\5\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}}{\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \frac{6}{12}\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} + \frac{6}{4}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}$$

18. (a)

$$U^TU = \begin{bmatrix}1/\sqrt{10} & -3/\sqrt{10}\end{bmatrix}\begin{bmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{bmatrix} = \begin{bmatrix}1/10+9/10\end{bmatrix} = \begin{bmatrix}1\end{bmatrix}$$

$$UU^T = \begin{bmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{bmatrix}\begin{bmatrix}1/\sqrt{10} & -3/\sqrt{10}\end{bmatrix} = \begin{bmatrix}1/10 & -3/10\\-3/10 & 9/10\end{bmatrix}$$

(b)

$$\operatorname{proj}_W\mathbf{y} = \frac{\begin{bmatrix}7\\9\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{bmatrix}}{\begin{bmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{bmatrix}}\begin{bmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{bmatrix} = \frac{-20/\sqrt{10}}{1}\begin{bmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix}$$

$$(UU^T)\mathbf{y} = \begin{bmatrix}1/10 & -3/10\\-3/10 & 9/10\end{bmatrix}\begin{bmatrix}7\\9\end{bmatrix} = \begin{bmatrix}-20/10\\60/10\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix}$$

19. By Theorem 6.8, the projection of $\mathbf{u}_3$ onto Span$\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$\hat{\mathbf{u}}_3 = \frac{\mathbf{u}_3\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_3\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}{\begin{bmatrix}1\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}}\begin{bmatrix}1\\1\\-2\end{bmatrix} + \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}5\\-1\\2\end{bmatrix}}{\begin{bmatrix}5\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}5\\-1\\2\end{bmatrix}}\begin{bmatrix}5\\-1\\2\end{bmatrix}$$

$$= \frac{-2}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix} + \frac{2}{30}\begin{bmatrix}5\\-1\\2\end{bmatrix} = \begin{bmatrix}0\\-2/5\\4/5\end{bmatrix}$$

and a vector orthogonal to this space is

$$\mathbf{z} = \mathbf{u}_3 - \hat{\mathbf{u}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -2/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \\ 1/5 \end{bmatrix}$$

Exercises 6.4 (p. 402)

Assignment: Do #17, 18, 4, 5, 8, 11, 15

17. (a) False. If $c = 0$, the new set won't be a basis.

(b) True. (p. 399)

(c) True. (p. 401)

18. (a) False. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal *and* has no nonzero vectors, then it will form a basis for W.

(b) True.

(c) True. (Theorem 6.12, p. 400)

4. By Gram-Schmidt, take

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$$

and take

$$\mathbf{v}_2 = \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} - \frac{\begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}}{\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}} \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} - \frac{-100}{50} \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} + \begin{bmatrix} 6 \\ -8 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis.

5. Take

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\mathbf{v}_2 = \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} - \frac{36}{18} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}$$

and $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis.

8. Normalizing the basis from problem 4 gives:

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{3^2 + (-4)^2 + 5^2}} \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{50}} \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{50} \\ -4/\sqrt{50} \\ 5/\sqrt{50} \end{bmatrix}$$

and

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \frac{1}{\sqrt{3^2 + 6^2 + 3^2}} \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{54}} \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{54} \\ 6/\sqrt{54} \\ 3/\sqrt{54} \end{bmatrix}$$

11. Using Gram-Schmidt,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} - \frac{-5}{5} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

and finally

$$\mathbf{v}_3 = \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} - \frac{20}{5} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{-12}{36} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5-4+1 \\ -4+4+0 \\ -3+4+1 \\ 7-4-1 \\ 1-4+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

giving a basis

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} \right\}$$

15. We must normalize the basis from problem 11 to produce an orthonormal basis for the columns of Q. This lengths of the vectors are $\sqrt{5}$, $\sqrt{36} = 6$, and $\sqrt{16} = 4$ respectively, so our orthonormal basis is:

$$\left\{ \begin{bmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \\ -1/\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \right\}$$

giving a Q matrix of

$$Q = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}$$

and an R matrix of

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{5} & -5/\sqrt{5} & 20/\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

giving the QR factorization.