Homework Set #10 Solutions

Exercises 2.9 (p. 174)

Assignment: Do #30, 32, 33, 34, 35, 36, 39, 40, 41, 42

30. We need to determine what weights c_1, c_2 satisfy $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \mathbf{x}$. In other words, we need to reduce the following augmented matrix:

$$\begin{bmatrix} 1 & -3 & -9 \\ -3 & 5 & 7 \end{bmatrix} \xrightarrow{R2 \to R2+3R1} \begin{bmatrix} \boxed{1} & -3 & -9 \\ 0 & -4 & -20 \end{bmatrix}$$
$$\qquad\Uparrow \qquad\qquad\qquad\qquad\qquad \Uparrow$$

$$\xrightarrow{R2 \to -\frac{1}{4}R2} \begin{bmatrix} \boxed{1} & -3 & -9 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{R1 \to R1+3R2} \begin{bmatrix} 1 & 0 & 6 \\ 0 & \boxed{1} & 5 \end{bmatrix}$$
$$\qquad\qquad\qquad\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

and the coordinate vector is the unique solution:

$$[\mathbf{x}]_{\mathfrak{B}} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

32. As in problem 30, we reduce the augmented matrix:

$$\begin{bmatrix} -3 & 7 & -8 \\ 1 & 5 & -1 \\ -4 & -6 & -3 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 5 & -1 \\ -3 & 7 & -8 \\ -4 & -6 & -3 \end{bmatrix} \xrightarrow{R2 \to R2+3R1} \begin{bmatrix} 1 & 5 & -1 \\ 0 & 22 & -11 \\ -4 & -6 & -3 \end{bmatrix}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\Uparrow \qquad\qquad\qquad\qquad\qquad \Uparrow$$

$$\xrightarrow{R3 \to R3+4R1} \begin{bmatrix} 1 & 5 & -1 \\ 0 & 22 & -11 \\ 0 & 14 & -7 \end{bmatrix} \xrightarrow{R3 \to R3-\frac{7}{11}R2} \begin{bmatrix} \boxed{1} & 5 & -1 \\ 0 & 22 & -11 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\qquad\qquad\qquad\qquad\Uparrow \qquad\qquad\qquad\qquad\qquad \Uparrow$$

$$\xrightarrow{R2 \to \frac{1}{22}R2} \begin{bmatrix} \boxed{1} & 5 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - 5R2} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & \boxed{1} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$
$$\qquad\qquad\qquad\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

and the unique solution gives the coordinate vector

$$[\mathbf{x}]_{\mathfrak{B}} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

33. By the Rank Theorem $\dim \operatorname{Col} A + 4 = 7$, so $\dim \operatorname{Col} A = 3$.

34. By the Rank Theorem $\dim \operatorname{Col} A + 2 = 5$, so $\dim \operatorname{Col} A = 3$.

35. Since $\dim \operatorname{Nul} A = 2$, $\operatorname{rank} A + 2 = 6$, so $\operatorname{rank} A = 4$.

36. Since $\operatorname{rank} A + 3 = 5$, we have $\operatorname{rank} A = 2$.

39. Obviously, the columns of A span $\operatorname{Col} A$ by the definition of $\operatorname{Col} A$. If they are also linearly independent, then by the definition of "basis", they are a basis for $\operatorname{Col} A$.

40. By the same reasoning as in problem 39, note that these vectors obviously span $H = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. If the set is also linearly independent, then by definition it is a basis for H.

41. Since A is invertible, its indexed set of columns $\mathfrak{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is linearly independent. Moreover, $\mathbf{u}$ is the unique solution to $A\mathbf{x} = \mathbf{b}$. That is,

$$u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n = \mathbf{b}$$

is the unique representation of $\mathbf{b}$ as a linear combination of the vectors $\mathfrak{A}$. By definition,

$$[\mathbf{b}]_{\mathfrak{A}} = \mathbf{u}$$

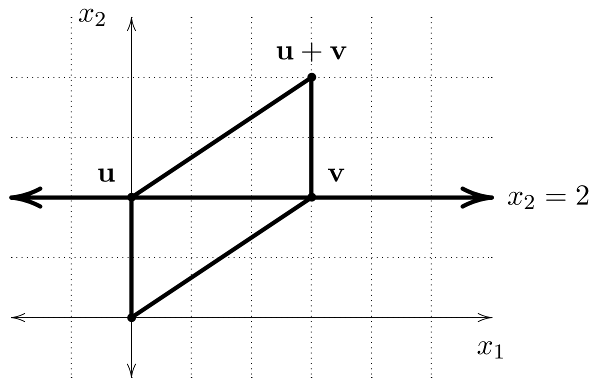
is the coordinate vector for $\mathbf{b}$ relative to $\mathfrak{A}$.

42. Obviously, the p columns of A span their own column space (by the definition of "column space"). If the column space is p-dimensional, then by the Basis Theorem, these p columns must be linearly independent.

Exercises 4.1 (p. 217)

Assignment: Do #4, 12

4. The following diagram shows how two points $\mathbf{u}$ and $\mathbf{v}$ on the line $x_2 = 2$ sum to a point $\mathbf{u}+\mathbf{v}$ not on the line.



12. We can write each such vector in the form

$$\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = \begin{bmatrix} s \\ s \\ 2s \\ 0 \end{bmatrix} + \begin{bmatrix} 3t \\ -t \\ -t \\ 4t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}, \quad s,t \in \mathbb{R}$$

Clearly, W is the span of these two vectors, and so it is a subspace of $\mathbb{R}^4$.

Exercises 4.2 (p. 228)

Assignment: Do #25, 26, 8, 10, 16

25. (a) True. (p. 220)
(b) False. It is a subspace of $\mathbb{R}^n$.
(c) True. (p. 227)
(d) False. If $A\mathbf{x} = \mathbf{b}$ is consistent *for every* $\boldsymbol{b}$, then $\operatorname{Col} A$ is $\mathbb{R}^m$.
(e) True. (p. 227)
(f) True. (p. 223)

26. (a) True. (p. 221)
(b) True. (p. 223)

(c) False. $\operatorname{Col} A$ has nothing to do with solutions.

(d) True. (p. 227)

(e) True. (p. 227)

(f) We didn't cover this; I don't expect you to know it. (It's true, though: see Example 9 on p. 228.)

8. No, it is not a vector space. Note that it does not contain the zero vector, for one thing.

10. Yes, it is a vector space. It is the set of all solutions to the system of equations given (in unknowns a, b, c, and d). However, this system can be rewritten as

$$\begin{cases} a - 3b - c = 0 \\ a + b + c - d = 0 \end{cases}$$

which is a homogeneous system. The set of all solutions to a homogeneous system is always a vector space.

16. Since we may write

$$\begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} = \begin{bmatrix} b \\ 2b \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -c \\ c \\ 5c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \\ -4d \\ d \end{bmatrix} = b\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix}, \quad b, c, d \in \mathbb{R}$$

This is clearly the span of the columns of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercises 4.3 (p. 237)

Assignment: Do #21, 22, 2, 6, 12, 20, 24

21. (a) False. A set consisting of one vector is linearly dependent if it is the zero vector. Otherwise, it is linearly independent.

(b) False. We need the set to be linearly independent, too.

(c) True. By the Invertible Matrix Theorem, the columns are linearly independent and span $\mathbb{R}^n$, so they are a basis for $\mathbb{R}^n$ by definition.

(d) False. A basis is a spanning set made as small as possible (by eliminating any vectors that are just linear combinations of other vectors in the set).

(e) False. EROs don't affect the linear dependence relations among the columns.

22. (a) False. It needs to span H, too.

(b) True. (Theorem 4.5, p. 234)

(c) True. (p. 236)

(d) False. The method always works, if used correctly.

(e) False. The pivot columns of A form a basis for $\operatorname{Col} A$. Usually the pivot columns of B won't work.

2. Since this set contains the zero vector, it is linearly dependent and not a basis. It does not span $\mathbb{R}^3$, since no vector with different first and third elements can be created by linear combinations of these vectors. (Alternatively, you could put the vectors into the columns of a matrix and show that the matrix doesn't have a pivot position in every row.)

6. These vectors are linearly independent (because they are not scalar multiples of each other), but they fail to span $\mathbb{R}^3$. Two linearly independent vectors span a plane (through the origin) in $\mathbb{R}^3$, not all of $\mathbb{R}^3$. (Alternatively, if you put them into the columns of a matrix, the matrix could have at most two pivot positions and couldn't have a pivot position in every one of its three rows.)

12. This is equivalent to the homogeneous system of one equation in two unknowns:

$$\{5x - y = 0$$

Putting the augmented matrix into reduced echelon form:

$$\begin{bmatrix} 5 & -1 & 0 \end{bmatrix} \xrightarrow{R1 \to \frac{1}{5} R1} \begin{bmatrix} 1 & -1/5 & 0 \end{bmatrix}$$

we see that the general solution is

$$\begin{cases} x = \frac{1}{5} y \\ y \text{ free} \end{cases}$$

giving a vector parametric form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{5} y \\ y \end{bmatrix} = y \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}, \quad y \text{ free}$$

That is, the set $\{(1/5, 1)\}$ is a basis for this line.

More directly, you could have noted that this was a line through the origin. Therefore, any nonzero vector contained in the line would serve as a basis for the line.

20. Since $\mathbf{v}_1 = 3\mathbf{v}_2 - 5\mathbf{v}_3$, by the Spanning Set Theorem, the set $\{\mathbf{v}_2, \mathbf{v}_3\}$ still spans H. However, these two vectors are linearly independent since they are not scalar multiples of each other. Therefore, $\{\mathbf{v}_2, \mathbf{v}_3\}$ will serve as a basis. (In fact, by the same reasoning, any two of the three vectors will serve as a basis.)

24. If we constructed the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$, it would have linearly independent columns. By the Invertible Matrix Theorem, its columns would span all of $\mathbb{R}^n$. Therefore, by definition, its columns would form a basis for $\mathbb{R}^n$.

Exercises 4.4 (p. 248)

Assignment: Do #2, 4, 6, 8, 20

2. The vector x is the linear combination of the vectors in $\mathfrak{B}$ using weights given by $[x]_{\mathfrak{B}}$, so

$$\mathbf{x} = 8\begin{bmatrix} 4 \\ 5 \end{bmatrix} - 5\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

4. As in problem 2,

$$\mathbf{x} = -4\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} - 7\begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$$

6. This is harder, since we must reduce the augmented matrix:

$$\begin{bmatrix} 1 & 5 & 4 \\ -2 & -6 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 + 2R1} \begin{bmatrix} \boxed{1} & 5 & 4 \\ 0 & 4 & 8 \end{bmatrix} \xrightarrow{R2 \to \frac{1}{4}R2} \begin{bmatrix} \boxed{1} & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R1 \to R1 - 5R2} \begin{bmatrix} 1 & 0 & -6 \\ 0 & \boxed{1} & 2 \end{bmatrix}$$

The coordinate vector is the unique solution

$$[\mathbf{x}]_{\mathfrak{B}} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

8. As in problem 6,

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{bmatrix} \xrightarrow{R3 \to R3 - 3R1} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 2 & -1 & -5 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R2} \begin{bmatrix} \boxed{1} & 2 & 1 & 3 \\ 0 & \boxed{1} & -1 & -5 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{R1\to R1-R3} \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & \boxed{1} & -1 & -5 \\ 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{R2\to R2+R3} \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 5 \end{bmatrix} \xrightarrow{R1\to R1-2R2} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 5 \end{bmatrix}$$

Again, the coordinate vector is the unique solution

$$[\mathbf{x}]_{\mathfrak{B}} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

20. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ is linearly dependent, then there's a linear dependence relation

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + d_4\mathbf{v}_4 = \mathbf{0}$$

for some $d_1, d_2, d_3, d_4 \in \mathbb{R}$ that aren't all zero. If $\mathbf{w}$ is some linear combination of these vectors:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{w}$$

then observe that

$$\begin{aligned}
&(c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + (c_3 + d_3)\mathbf{v}_3 + (c_4 + d_4)\mathbf{v}_4 \\
&\quad = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4) \\
&\qquad + (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + d_4\mathbf{v}_4) \\
&\quad = \mathbf{w} + \mathbf{0} = \mathbf{w}
\end{aligned}$$

Since the d_i aren't all zero, the weights $c_1, \ldots, c_4$ and $c_1 + d_1, \ldots, c_4 + d_4$ give different linear combinations equal to the same vector $\mathbf{w}$.

Exercises 4.5 (p. 255)

Assignment: Do #19, 20, 9

19. (a) True. (p. 254)

(b) True. (Example 2, p. 252)

(c) We didn't cover this; I don't expect you to know it. It happens to be false, though. On p. 251, the author mentions that $\dim \mathbb{P}_n$ is $n + 1$, not n.

(d) False. We need S to have size n before we can apply the Basis Theorem.

(e) True. Some subset of the set is a basis, and its size will be no larger than p. Therefore, $\dim V \leq p$. Now, we may apply Theorem 4.9.

20. (a) False. $\mathbb{R}^2$ isn't a subset of $\mathbb{R}^3$.

(b) False. The number of *free* variables is the dimension of $\operatorname{Nul} A$.

(c) We didn't cover this; I don't expect you to know it. It's false. A vector space is infinite-dimensional if it can't be spanned by a finite set.

(d) False. Again, this only works if S has n vectors in it.

(e) True. (p. 253)

9. This is the set

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$

Since we can write this in vector parametric form as

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_1,\ x_2 \text{ free}$$

the span of two, linearly independent vectors, they form a basis for the set, and so it must have dimension 2.

Exercises 5.1 (p. 302)

Assignment: Do #21, 22, 2, 4, 16, 20, 23, 24, 29, 31

21. (a) False. The vector must be *nonzero*, too.

(b) True. (p. 301)

(c) True. (p. 298)

(d) True. (p. 297)

(e) False. Row reduction is useful for finding eigen*vectors* (or eigenspaces) for particular eigenvalues, but it isn't much help for finding eigen*values*.

22. (a) False. The vector must be *nonzero*, or it isn't an eigenvector.

(b) False. It's true that eigenvectors that correspond to distinct eigenvalues are linearly independent. But the converse isn't true: linearly independent eigenvectors don't have to correspond to distinct eigenvalues.

(c) We didn't cover steady-state vectors; I don't expect you to know it. It's true, though: a steady-state vector is a nonzero vector that satisfies $A\mathbf{v} = \mathbf{v}$, so its an eigenvector for eigenvalue 1.

(d) False. That's only true for a triangular matrix.

(e) True. (p. 298)

2. For a 2×2 matrix, the easiest way to check if λ is an eigenvalue is to calculate $\det(A - \lambda I)$. If this is zero then the matrix is singular. Then, by the Invertible Matrix Theorem, $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution so the equation $A\mathbf{x} = \lambda\mathbf{x}$ has a nontrivial solution so by definition λ is an eigenvalue of the matrix.

Now,

$$\det(A - (-2)I) = \det\left(\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\right) = \det\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} = 0$$

Therefore, *yes*, -2 is an eigenvalue for A.

4. For this problem, just calculate the matrix-vector product and see if it's a scalar multiple of the original vector or not:

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 2(-1+\sqrt{2})+1(1) \\ 1(-1+\sqrt{2})+4(1) \end{bmatrix} = \begin{bmatrix} -1+2\sqrt{2} \\ 3+\sqrt{2} \end{bmatrix}$$

This is now a little tricky, since—with those square roots in there—it's not totally obvious whether or not the right-hand side is a scalar multiple of the original vector. However, note that, if it is, it must be the product of the original vector by $3+\sqrt{2}$ (in order for the second entry to come out right). But,

$$(3+\sqrt{2})\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} (3+\sqrt{2})(-1+\sqrt{2}) \\ 3+\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2}+1 \\ 3+\sqrt{2} \end{bmatrix}$$

It's closer than we might have expected, but that's not quite a match, so the given vector is *not* an eigenvector of the matrix.

16. Finding a basis for the eigenspace involves finding a basis for the solution set of the homogeneous equation $(A - 4I)\mathbf{x} = \mathbf{0}$. The coefficient matrix is:

$$A - 4I = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reducing the augmented matrix:

$$\begin{bmatrix} -1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \to R2+R1} \begin{bmatrix} -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R3 \to R3+R2} \begin{bmatrix} \boxed{-1} & 0 & 2 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R2 \to -R2} \begin{bmatrix} -1 & 0 & 2 & 0 & 0 \\ 0 & \boxed{1} & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to -R1} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & \boxed{1} & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

gives a general solution

$$\begin{cases} x_1 = 2x_3 \\ x_2 = 3x_3 \\ x_3,\ x_4 \text{ free} \end{cases}$$

and vector parametric form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 3x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_3,\ x_4 \text{ free}$$

Therefore, the eigenspace has basis

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

20. One eigenvalue is really easy to find. The columns of A are linearly dependent, so A is singular. Therefore, it has an eigenvalue 0 by the first Theorem on p. 301. To get an eigenvector for this eigenvalue, all we need is a nontrivial solution to $A\mathbf{x} = \mathbf{0}$, but that's really the same as finding a linear dependence relation between the columns. For example, because the first column minus the second is zero, that means:

$$1\mathbf{a}_1 + (-1)\mathbf{a}_2 + 0\mathbf{a}_3 = \mathbf{0}$$

and so $(1, -1, 0)$ is a nontrivial solution to $A\mathbf{x} = \mathbf{0}$. Since the first and third columns are equal, too, that gives another nontrivial solution $(1, 0, -1)$. And these two nontrivial solutions are linearly independent (because two vectors that aren't scalar multiples of each other are linearly independent). Other sets of linearly independent eigenvectors are possible here, too. For example, $\{(1, 1, -2), (2, -1, -1)\}$ is another linearly independent set of eigenvectors for the eigenvalue 0.

There's only one other eigenvalue, and it's not too hard to find either. For a general vector $\mathbf{x}$, we have

$$A\mathbf{x} = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_1 + 5x_2 + 5x_3 \\ 5x_1 + 5x_2 + 5x_3 \\ 5x_1 + 5x_2 + 5x_3 \end{bmatrix}$$

so A times any vector yields a vector all of whose elements are the same. A good candidate for an eigenvector would be a vector all of whose elements were the same—multiplied by A, it would become another vector all of whose elements are the same, and that would be scalar multiple of the original vector. So, for example:

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} = 15 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

That is, 15 is an eigenvalue with eigenvector $(1, 1, 1)$.

23. If a 2×2 matrix had three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, then by Theorem 5.2, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ would be linearly independent. But a set of three vectors in $\mathbb{R}^2$ *can't* be linearly independent by Theorem 1.8. Therefore, a 2×2 matrix can have at most two distinct eigenvalues.

The argument for general n is no different. If you have more than n distinct eigenvalues, you'll have a linearly independent set of more than n eigenvectors by Theorem 5.2. But no set of more than n vectors in $\mathbb{R}^n$ can be linearly independent.

24. Triangular matrices are easiest to deal with. For example, the matrix

$$A = \begin{bmatrix} 5 & 0 \\ 1 & 5 \end{bmatrix}$$

has only one distinct eigenvalue (namely 5) by Theorem 5.1 In fact, the 2×2 identity matrix is a good one:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It's triangular, too, so its eigenvalues are the entries on its main diagonal, but the only distinct value on the main diagonal is 1, so 1 is the only eigenvalue of the identity matrix.

29. Problem 20 is a special case of this kind of matrix. (Note that the row sums of that matrix are all equal to 15). That's a good hint.

For a general $n \times n$ matrix A, note that multiplying A by the size n vector $\mathbf{v} = (1, \ldots, 1)$ all of whose entries are 1 gives a vector $A\mathbf{v}$ whose entries are the row sums of the corresponding rows of A. (The first entry of $A\mathbf{v}$ is the sum of the entries of A's first row, the second is the sum of the second row, and so on.) If all the row sums are equal to some number s, then $A\mathbf{v}$ is a vector all of whose entries are s (which is equal to s times $\mathbf{v}$). In symbols, we have

$$A \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = s \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

so by definition s is an eigenvalue of A.

31. Note that every vector *on* the line of reflection doesn't move. That is for any vector $\mathbf{v}$ on the line, we have $A\mathbf{v} = \mathbf{v}$. So, A has an eigenvalue 1, and its eigenspace consists of the line of reflection.

Also, every vector on the line L through the origin *perpendicular* to the line of reflection flips across to other side (that is, gets taken to its negative). That is, for any vector $\mathbf{w}$ on the line L, we have $A\mathbf{w} = -1\mathbf{w}$. So, A has another eigenvalue -1 whose eigenspace is the line L.

Exercises 5.2 (p. 311)

Assignment: Do #21, 22, 2, 4, 12, 16

21. (a) False. That's true only if A is triangular.

(b) False. Row interchange changes the sign, and row scaling scales the determinant.

(c) True. (Theorem 5.3, p. 307)

(d) False. If $\lambda + 5$ is a factor then $\lambda = -5$ is the corresponding eigenvalue (because that's the value of λ that makes the factor, and so the whole polynomial, evaluate to zero).

22. (a) True. (p. 306)

(b) False. $\det A^T = \det A$

(c) True. (p. 308)

(d) False. (p. 309)

2. The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = (25 - 10\lambda + \lambda^2) - 9 = \lambda^2 - 10\lambda + 16$$

We can factor the polynomial to find the roots of the characteristic equation:

$$0 = \lambda^2 - 10\lambda + 16 = (\lambda - 2)(\lambda - 8)$$

so $\lambda = 2$ and $\lambda = 8$ are the eigenvalues of A.

4. The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -3 \\ -4 & 3-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda) - 12 = (15 - 8\lambda + \lambda^2) - 12 = \lambda^2 - 8\lambda + 3$$

The characteristic equation is:

$$0 = \lambda^2 - 8\lambda + 3$$

Factoring doesn't seem to help, so we can use the quadratic formula:

$$\begin{aligned} \lambda &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{8 \pm \sqrt{64 - 4(1)(3)}}{2} \\ &= 4 \pm \frac{\sqrt{52}}{2} \end{aligned}$$

That is, the two eigenvalues are $4 + \sqrt{52}/2$ and $4 - \sqrt{52}/2$.

12. The characteristic polynomial is given by

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

A cofactor expansion across the third row gives:

$$\det(A - \lambda I) = 0 - 0 + (2-\lambda) \begin{vmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda)(4-\lambda)$$

Note that this means the characteristic equation has roots 2, -1, and 4, so those are the three eigenvalues of A.

16. Since this is a triangular matrix, its eigenvalues are just the diagonal elements, 5, -4, 1, and 1. Note that 1 is listed twice; it has a multiplicity of two.

Exercises 5.3 (p. 319)

Assignment: Do #21, 22, 2, 6, 8, 20

21. (a) False. D must be diagonal.

(b) True, though the phrasing is confusing. If $\mathbb{R}^n$ has a basis consisting of some of A's eigenvectors then this basis must consist of n linearly independent vectors. In other words, A has n linearly independent eigenvectors. By Theorem 5, A must be diagonalizable.

(c) False. If the eigenvalues are distinct, then A is diagonalizable. Otherwise, we have to study the dimensions of the eigenspaces and compare them to the multiplicities of the eigenvalues before we can decide.

(d) False. Example 5, p. 317 gives a diagonalizable matrix that is not invertible (because 0 is one of its eigenvalues).

22. (a) False. They need to be linearly independent eigenvectors.

(b) False. Example 3, p. 315 gives an example of a diagonalizable 3×3 matrix with only two distinct eigenvalues.

(c) True! See the proof of Theorem 5.5 on p. 315. For any P (even one that isn't invertible), AP is given by formula (1) and PD is given by formula (2). If $AP = PD$, then $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \ldots, A\mathbf{v}_n = \lambda_n\mathbf{v}_n$, and any nonzero columns $\mathbf{v}_k$ are eigenvectors by definition.

(d) False. Example 4, p. 317 gives an invertible matrix that is not diagonalizable. (The matrix is invertible because 0 isn't an eigenvalue.)

2. The inverse of P is given by the formula:

$$P^{-1} = \frac{1}{2(5) - (-3)(-3)} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

and A^4 is given by the trick from Example 2, p. 314:

$$\begin{aligned} A^4 = PD^4P^{-1} &= \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1^4 & 0 \\ 0 & (\frac{1}{2})^4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \\ &= \left(\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{16} \end{bmatrix} \right) \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{16} \\ -3 & \frac{5}{16} \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{151}{16} & \frac{90}{16} \\ \frac{-225}{16} & \frac{-134}{16} \end{bmatrix} \end{aligned}$$

6. The eigenvalues are 5, 5, and 4.

The eigenvalue 5 that appears in the first and second diagonal elements of D has eigenvectors from the first two columns of P. (These columns will be linearly independent by Theorem 5.5, and the dimension of the eigenspace associated with 5 will be equal to its multiplicity 2 by

Theorem 5.7. Therefore, these two columns will form a basis. This is always what happens when we're working with a diagonalization of A.) Therefore,

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for the eigenspace associated with eigenvalue 5.

The eigenvalue 4 appears as the third diagonal element of D, so its eigenvector is P's third column. That is,

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

is a basis for the eigenspace associated with eigenvalue 4.

8. This is a triangular matrix, so its eigenvalues are 5 and 5: that is, it has one distinct eigenvalue 5 with multiplicity 2. We now find a basis for the eigenvalue 5 by finding a basis for $\mathrm{Nul}(A - 5I)$. Since $A - 5I = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, the augmented matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is already in reduced echelon form. Its general solution is

$$\begin{cases} x_2 = 0 \\ x_1 \text{ free} \end{cases}$$

with vector parametric form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_1 \text{ free}$$

In other words, the eigenspace associated with eigenvalue 5 has dimension 1. Since this doesn't match the eigenvalue's multiplicity (of 2), by Theorem 5.7, the matrix is not diagonalizable.

20. Again, the matrix is triangular, so it has eigenvalues 4, 4, 2, and 2.

The eigenspace for eigenvalue 2 is the null space of $A - 2I$. We can calculate a basis by reducing the augmented matrix:

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{R2 \to \frac{1}{2}R2}]{\substack{R4 \to R4 - \frac{1}{2}R2 \\ R1 \to \frac{1}{2}R1}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

That is, the general solution is

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3,\ x_4 \text{ free} \end{cases}$$

with vector parametric form

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_3,\ x_4 \text{ free}$$

This gives us two linearly independent eigenvectors as a basis for this eigenspace.

Moving on to eigenvalue 4. Its eigenspace is the null space of $A - 4I$, so we calculate its basis by reducing the augmented matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{bmatrix} \xrightarrow[R2 \to -\frac{1}{2}R2]{\substack{R1 \leftrightarrow R4 \\ R2 \leftrightarrow R3}} \begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, the general solution is

$$\begin{cases} x_1 = 2x_4 \\ x_3 = 0 \\ x_2,\ x_4 \text{ free} \end{cases}$$

with vector parametric form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_4 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_2,\ x_4 \text{ free}$$

This gives us two linearly independent eigenvectors as a basis for this eigenspace.

In total, we have four linearly independent eigenvectors, so by Theorem 5.5, the matrix A is diagonalizable. We'll put the eigenvalues into a diagonal in the following order:

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

So, in P, we'll have to put our four eigenvectors into the columns in the corresponding order: first the two eigenvectors for eigenvalue 2, then the two eigenvectors for eigenvalue 4:

$$P = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$